# Equilibria in concave non-cooperative games and their applications in smart energy allocation

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**Abstract.** Game theory is often applied to modeling interactions of non-cooperative decision makers. Such interaction appear, among others, in the case of energy management. In this context we formulate the problem of energy allocation for a group of electric vehicles in a smart grid. Subsequently, we formulate a game-theoretic model of interactions of agents controlling vehicle charging schedules. An algorithm for computing pure Nash equilibrium in such game is presented. Moreover, we introduce a solver, which is specifically designed to find equilibria in concave games. The core of the proposed solver is based on the primal-dual interior-point method for nonlinear programming. Experimental results of applying the solver are compared with a centralized solution.

*Keywords*: game theory, smart energy, demand side management, convex optimization, charging electric vehicles, energy allocation

## 1 Introduction

Electric vehicles have a long history [1], but only now their popularity is rising. This is, among others, due to the availability of cheaper and more ecological energy from renewable power sources. Electric vehicles are pollution free and "almost silent." Their range and speed is often not worse than those of conventional models (especially in cities, where traveled distances are limited and speed is reduced by the traffic). Their main limitations are the capacity and the physical properties of batteries, which tend to be heavy, not very durable and, constrained by the speed of charging. Furthermore, the infrastructure for battery-recharge (or exchange) is still insufficient. However, there are many ongoing country-level projects that fund construction of charging stations, for example the UK government dedicated £37 million to building charging stations [6]. Furthermore, the pressure on decreasing the emission levels (in particular in the cities) is helping speed-up the development of infrastructure for electric cars.

In this context, need for solving the problem of charging a group of electric vehicles, is no longer a futuristic one. Note that, a sudden peak in the power grid, caused by multiple electric vehicles that are to be charged, has to be quickly compensated by the existing power production. While currently, charging electric vehicles takes only a fraction of the power of the network, with the increasing number of cars this fraction will increase fast. This might cause serious power peaks, as fast-charging vehicles can take large amounts of power in a very short time, which might destabilize the currently existing power grid. Control over maximum power usage can help avoiding such power peaks. Therefore, it can be assumed that charging stations will have limitations on the amount of power available to connected cars. Optimizing discharge of available power, becomes important, to allow owners to "optimally" use their vehicles. Furthermore, stable power consumption allows contracting energy for long term. Such long-term deals are much more beneficial both both the supplier (planning power production) and the consumer (who pays less). Note that deviations from the contracted power consumption are additionally charged.

Therefore, we consider charging a group of electric vehicles in a charging station placed within a smart grid. Following one of the major approaches to the development of smart grids, we assume that electric cars are controlled by software agents (one per vehicle) [3, 12]. One of the main ideas underlying the smart grid is that vehicles can be flexibly charged, according to their current needs (e.g. the distance of the next anticipated travel or the time when the car/motorbike will be needed again). In this scenario, a game-theoretic model of interactions between agents controlling the process of charging the vehicles can be applied. Here, allocation decisions are made independently by each agent in a distributed manner [13, 18].

### 1.1 Detailed problem description

Problems faced by the power grid are varied: balancing power, power peaks, failures, unpredictability of usage and of production, etc. Here, we consider a small subclass of the Demand Side Management (DSM) problem: allocation of energy to a group of electric vehicles to be charged at a charging station, placed within a smart grid environment. This problem concerns distribution of scarce resources and thus it is assumed that agents representing individual vehicles are selfish, as cooperation requires existence of common goals and communication. While there can be a common goal (e.g. shaving the power peaks or reducing the cost of electricity by signing a long-term contract with the provider), such goals are placed within policies of the charging station, and their management is out of the scope of our work. Therefore, agents representing cars compete for power and a game theoretic approach becomes appropriate.

Let us now describe our use case scenario. We assume that a "charging station" has a limited number of charging slots and a limit on the total amount of power it can draw at one time. Such limit is the effect of long-term deals with the energy suppliers as well as the actual power infrastructure within and outside of the station. Each car has a software agent installed, which is responsible, among others, for vehicle's battery charge/exchange planning. These plans are to match the user's needs. Note that the "special" situation, when batteries are to be exchanged/replaced, is omitted from the current contribution. The charging station allows the vehicles, to charge with an appropriate speed (related to the throughput of the slot and battery limitations) in a sequence of fixed time periods. Car agents define their strategies concerning how much power needs to be charged in a given time period for "their vehicle." The minimal goal is to reach the minimal required charge level, e.g. to complete the next trip (the ultimate goal is to reach the total capacity of the battery). If there are too many vehicles with large power requirements, it is not possible to charge them concurrently at high rates. As a result, the actual charging speeds are decided by the charging station; when the vehicles with their proclaimed demands arrive and connect to the charging slots. We assume that all vehicles arrive once per day (we plan to relax this restriction in the future), and are supposed to be left at the charging station for up to a fixed number of hours (e.g., 10 hours at a time).

Here, it is assumed that agents in vehicles arriving at the station for charging are not aware of other agents demands. Thus charging plans need to be negotiated among agents representing all vehicles. This can be interpreted as a non-cooperative game. Here, we omit a situation when charging schedule negotiations involve also selection of the charge station. In other words, the individual charging schedule involves its power needs and the limited power output of the selected station. Consequently, in order to achieve the highest charging efficiency, agents should construct charging schedules, which correspond to the Nash equilibria of an allocation game (defined in Section 4). Proposed approach can be extended to consider the changing prices of electric energy, battery exchange, number of charging periods during the day, selection of the charging stations among these that are available, etc.).

The paper is organized as follows. In Section 2 we present an overview of the related works. We follow, in Section 3, with the definition of the needed mathematical notation. Next, in Section 4, we formulate the decision making problem. Section 5 contains details of software developed to solve the problem. In Section 6 we summarize the results of an experimental study. Finally, in Section 7 we summarize the paper and outline future research directions.

## 2 Related Works

Considered problem is typically called the Demand Side Management (DSM) or the Demand Response (DR). A comprehensive review of the literature of this topic can be found in [3]. There are two main approaches to the solution of the DSM: (a) planning and scheduling power usage [10], and (b) dynamically shifting consumption towards a better moment [21]. Research in the DSM includes contributions from mathematics, game theory and social psychology [19].

In power management, some devices can be automatically delayed or interrupted. However, when an operated by a human device might be of little importance, the user might refuse to switch it off. Separately, power storage units allow compensating for sudden peaks of energy consumption and, consequently, may limit the daily variability of power use [22]. In considered problem, devices are electric cars equipped with batteries. Such batteries can, in theory, be used as a general power storage. However, their main use is to power the electric vehicle. Therefore, the key problem is to charge the car in a limited time, without overloading the power grid. Car batteries have relatively high capacity. Therefore, while a single car is not causing a big charging load, a large number of cars can result in a serious load for the grid. The effect of using plug-in electric vehicle (PHEV) on the power grid, including the influence on its stability, is discussed in [9]. Here, author analyses the charging patterns of batteries and shows that electric vehicle can be considered a Flexible AC Transmission System and can help improve the power quality in the energy network. In [11] the decentralized control method of charging electric vehicle is presented. In that work, the large number of electric cars is considered and the charging control goal is to shift the power usage by cars to off-peak time and, by doing that, reduce the cost of supping power supply. Separately, in [17], the state of the art concerning charging electric vehicles and its effect on power prices is considered. Authors show that the electric/hybrid vehicles are much cheaper on average. In [4, 24] the amount of power used during charging and the payoff were considered. However, other aspects, such as the order, speed and/or time of charging, were omitted. While these aspects can be simplified in theory (resulting in an easier model), in the actual power systems they have to be considered.

The analysis of battery operation, found in [22], considers the amount of charged or discharged power, in a time interval of a predefined length. This publication provides foundation for the game model, proposed in our paper.

### 2.1 Game-theoretic approach

Most of work in non-cooperative game theory concerns games with a finite sets of strategies. In such games decision makers choose among a predefined sets of actions. Here, only mixed-strategy Nash equilibria are guaranteed to exist (see, for instance, [5]). Furthermore, the complexity class PPAD (Polynomial Parity Arguments on Directed Graphs; [15]) captures the inherent combinatorial difficulty of this type of problems. Finally, it is conjectured that *no* polynomial-time algorithms exist for solving them.

Concave games, in contrast, are computationally less demanding. By allowing the decision makers to choose from a continuum of decisions, and by exploiting properties of payoff functions, it is possible to reach an equilibrium in polynomial time. It has to be stressed that such games still model decision making problems of practical importance. For instance, in a packet-based computer network, a sender may wish to select the transmission speed in a channel of limited capacity (shared with other transmissions) [8,20]. Financial institutions may select prices of their assets and expect yields depending on all prices of assets available on the market. Users of smart energy grids may use only some of their deices – when the supply of energy, for all users, is limited – and energy has to be shared.

Recently, we have developed a software package aimed at efficiently solving concave games. There exist a number of packages for convex programming, using highly efficient implementations of primal-dual interior-point method. Our work aims at providing a similar functionality for the non-cooperative game theory. The tool under development will allow easy description of the input, while efficiently computing the equilibria.

Here, note that a centralized solution can be found for the considered problem. However, it requires providing information about the level of battery charge, required battery level, and other data, which might be considered a violation of privacy by the owner of the vehicle. Furthermore, as was mentioned in [11], the owners of vehicles are reluctant to give away the control over the charging procedure. Furthermore, solutions where the agent of a car suggests strategies for charging its vehicle allow the system to consider special constraints (e.g. controlling the number of charge cycles) that might prolong the life of the battery.

## 3 Definitions

Strategic (mathematical) games are used to model situations of conflict (or cooperation) between two or more players. Each player decides on its *strategy* (also called *action*), and receives a *payoff*, which, in general, depends on strategies of other players. It is assumed that each player is rational, and wants to maximize its payoff. For more details, see [14]. Now, let $\mathcal{N} = \{1, \ldots, N\}$, $N \geq 2$, be the set of players. A *non-cooperative game* is defined by specifying sets of strategies $\{S_i\}_{i\in\mathcal{N}}$ and payoff functions $\{u_i\}_{i\in\mathcal{N}}$ that are to be maximized (alternatively, *cost functions* $c_i = -u_i$ can be defined, and the goal of each player would be to minimize them). The set $S_i$ is called the set of *pure strategies* of $i$th player, or its *strategy space*. Vector $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, where $x_i \in S_i$, is called *strategy profile* of the game, and consists of strategies $x_i$ of all players. Value of $u_i(\mathbf{x})$ defines the *payoff* of $i$th player, resulting from a strategy profile $\mathbf{x}$. The following notation is conventionally used for the strategy profile:

$$\mathbf{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N).$$

It denotes a vector of all strategies, except that of the $i$th player. Note that notation $\mathbf{x} = (x_i, \mathbf{x}_{-i})$ is often used to distinguish the $i$th player's strategy. Assuming that players make decisions *independently* and are characterized by *selfishness*, the best outcome of the game would be the one in which each player realizes the best response to all other players' strategies. A (*pure*) *Nash equilibrium* of a game with such strategy profile $\bar{\mathbf{x}}$ is:

$$\forall i \in \mathcal{N},\ \ \forall x_i :\ \ u_i(\bar{x}_i, \bar{\mathbf{x}}_{-i}) \geq u_i(x_i, \bar{\mathbf{x}}_{-i}).$$

If each player decided on a strategy $\bar{x}_i$, such that $\bar{\mathbf{x}}$ is a Nash equilibrium, then no player has an incentive to change its strategy, as such change is not going to improve its payoff. Thus, such strategy profile can be seen as the "socially best profiles." Therefore, non-cooperative, rational, selfish agents should prefer to use strategies resulting in a Nash equilibrium.

Let us now consider *concave* non-cooperative games. Here, the strategy of each player is a vector in the Euclidean space $\mathbf{x}_i \in \mathbb{R}^{m_i}$, $i = 1, \ldots, N$. Each strategy space set $S_i$ is a convex set. The payoff function $u_i$ is continuous in $\mathbf{x}$, and is concave in $\mathbf{x}_i$, for each fixed value $\mathbf{x}_{-i}$. Alternatively, consider cost functions $c_i$, continuous in $\mathbf{x}$, and convex in $\mathbf{x}_i$, for each fixed value $\mathbf{x}_{-i}$. It is well-known that pure Nash equilibrium always exists in concave games [16]. An equilibrium point is a solution of a system of nonlinear equations, similar to the Karush-Kuhn-Tucker (KKT) conditions, in standard optimization. Assume that the strategy space of $i$th player can be defined by a set of differentiable functions:

$$S_i = \{\mathbf{x}_i : h_{i1}(\mathbf{x}_i) \geq 0, h_{i2}(\mathbf{x}_i) \geq 0, \ldots, h_{ik_i}(\mathbf{x}_i) \geq 0\}.$$

Finally, in the considered game, an equilibrium point $\mathbf{x}$ must satisfy the feasibility conditions of all strategy spaces, i.e.:

$$\forall i \in \mathcal{N},\ \forall j \in \{1, \ldots, k_i\} : \quad h_{ij}(\mathbf{x}_i) \geq 0, \tag{1}$$

as well as the complementary slackness conditions:

$$\forall i \in \mathcal{N},\ \forall j \in \{1, \ldots, k_i\},\ \exists \lambda_{ij} \geq 0 : \quad \lambda_{ij} h_{ij}(\mathbf{x}_i) = 0, \tag{2}$$

and the stationarity conditions of Lagrange functions:

$$\nabla_{\mathbf{x}_i} u_i(\mathbf{x}) - \sum_{j=1}^{k_i} \lambda_{ij} \nabla_{\mathbf{x}_i} h_j(\mathbf{x}_i) = 0, \forall i \in \mathcal{N}. \tag{3}$$

## 4 Problem formulation

Let us now consider an optimization problem representing the vehicle charging scenario. Let $\mathcal{N}$ denote the set of electric vehicles (EVs), where $|\mathcal{N}| = N$. Each of them has an energy demand $D_n > 0$, as well as a battery capacity $C_n > 0$, $n = 1, \ldots, N$. The total charging time of all $N$ vehicles is divided into a fixed number of $T$ discrete intervals (e.g., 1-hour intervals). Each vehicle needs to formulate a charging plan $\mathbf{x}_n = [x_{n,1}, \ldots, x_{n,T}]^\intercal$, where $x_{n,t}$ is the $n$-th vehicle's requested charging rate for $t$-th time interval, while $x_{n,t}$ represents the requested speed of charging $n$-th vehicle's battery during $t$-th time interval. Observe that it is *not* assumed that vehicles arrive at the charging station at the same time, but the access time to the charging slots is discretized: a vehicle may set its requested rates to 0 for some of $T$ charging periods, which means that it does not have to be connected to the charging station then. In order for the $n$-th vehicle to satisfy its demand, it must receive the total energy allocation equal to $\sum_{t=1}^{T} x_{n,t}$, which must reach at least the amount of energy needed for the next expected travel (but it cannot be greater than the capacity of its battery):

$$\forall n \in \mathcal{N} \ \ D_n \leq \sum_{t=1}^{T} x_{n,t} \leq C_n. \tag{4}$$

Additionally, in order for a vehicle to be operational, it is required that its energy level never falls below a minimum energy reserve threshold. To assure this, for each charging interval $t$ there is a rate lower bound $L_{n,t}$ given by:

$$\forall n \in \mathcal{N} \;\; \forall t \in \{1, \dots, T\} \;\; x_{n,t} \geq L_{n,t}. \tag{5}$$

Note that values $L_{n,t}$ do not have to be positive, as we may allow, in a given time interval $t$, for discharging the battery (negative values of $x_{n,t}$ are interpreted as discharging rates). However, here we consider only the case when $L_{n,t} \geq 0$. Nevertheless, generalizes to include discharging. The reserve threshold is not explicitly given in the input data, as it is enough to provide values of $L_{n,t}$.

Although user agents may select any rate requests satisfying (4)–(5), the actual charging rate is allocated by the charging station, taking into account the total requestes from all $N$ vehicles. In each $t$-th time interval, each $n$-th vehicle receives a fraction $\rho_t$ of its requested rate $x_{n,t}$, where $\rho_t = f(\sum_{j=1}^{N} x_{j,t})$; the function $f : \mathbb{R} \to [0, 1]$ is a nonincreasing function of a total of requested rates. It is selected in order to prevent the station overcharge. In general, if the station has a fixed supply $S$ units of energy for one charging period $t$, then for the aggregate demand $d > S$, $f(d) < 1$ must be selected so that $\sum_{j=1}^{N} x_{j,t} f(d) < S$. The faster the function $f$ decreases, the more the station penalizes the aggregate demands that are too high. Here, we restrict $f$ to linear functions, leaving choice of other functions for future investigations. The total energy that the $n$-th vehicle receives from the station in the time period consisting of $T$-intervals is equal to:

$$u_n(\mathbf{x}_n) = \sum_{t=1}^{T} x_{n,t} f(\sum_{j=1}^{N} x_{j,t}). \tag{6}$$

Now, we can now define the following energy-allocation game. Let us assume that $\mathcal{N}$ players have feasible strategies defined as the set of all vectors $\mathbf{x}_n = [\mathbf{x}_1, \dots, \mathbf{x}_T]^\intercal$ satisfying (4)–(5). Here, the goal of each player is to maximize the payoff function defined as in (6). In other words, each player must select the charging rate resulting fastest charging, but must take into consideration the fact that requesting too high charging rate by many players will be penalized by the reduced energy flow from the station. Thus, each player should individually balance its request between fast charging and keeping charging rates low; to prevent the station overcharge (which would penalize all players).

## 5 Software solver

### 5.1 Representation of games

Let us now describe in more detail the software that we have developed for solving convex/concave non-cooperative games. To solve a game, we first need to pass it as an input to the solver. The developed software uses a relatively simple syntax, which is presented in Example 1, and stores game descriptions as a text file.

*Example 1.* Input file representing a simple two-player instance of the considered game.

```
N 2
S1 (x11, x12) {
  10 - x11 - x12
}
S1 {
  x11 + x12 - 5
}
S2 (x21, x22) {
  20 - x21 - x22
}
S2 {
  x21 + x22 - 10
}
P1 {
  set y = 1 - x11 - x21 - x12 - x22
  x11 * y + x12 * y
}
P2 {
  set y = 1 - x11 - x21 - x12 - x22
  x21 * y + x22 * y
}
```

Here, the first line defines the number of players, indicated by the integer after symbol `N` (two players in this case). Following are definitions of the strategy spaces of each player. A strategy space is defined in the form:

$$h_{ij}(\mathbf{x}_i) \geq 0,$$

where $h_{ij}$ is the $j$th constraint of the $i$th player's strategy space. User must provide formulas for $h_{ij}$, for each player, which is accomplished in constraint blocks, denoted by the symbol `S`, immediately followed by the index of the player. Names of player's decision variables must be given in parentheses before the first constraint block (and can be omitted in each subsequent block). The body of the function itself must be contained within brackets. In Example 1, there are two constraints defining the strategy space of *Player 1*: $h_{11}(\mathbf{x}_1) = 10 - x_{11} - x_{12}$, and $h_{12}(\mathbf{x}_1) = x_{11} + x_{12} - 5$, where $\mathbf{x}_1 = (x_{11}, x_{12})$.

Subsequently, the payoff functions are defined in function blocks, starting with the symbol `P`, followed by the index of the player. The value of the payoff can depend on all decision variables of all other players. Thus any subset of decision variables of all players may appear in the block defining payoff function.

The value of the last expression in each block is the payoff. Observe that computations can be simplified using `set` expressions, which define the auxiliary variables. For instance the variable `y` defined at the beginning of both payoff functions, above. Here, variable $\mathtt{y}$ appears multiple times in the second line of the payoff function, but the expression is evaluated only once.

The input syntax supports arithmetic operations on floating point numbers, as well as all standard mathematical functions (min/max, logarithms, exponentiation, trigonometric functions).

### 5.2 Optimization algorithm

The core solver is based on the primal-dual interior-point method from nonlinear programming. The method seeks to find a solution to the relaxed KKT conditions, which define a system of equations (1)–(3). Found solution approximates pure Nash equilibrium in the convex/concave non-cooperative game. By regulating the relaxation parameter one may obtain the approximation with an arbitrary accuracy (bounded only by the use of floating-point arithmetic). For each player $n \in \mathcal{N}$ we can formulate the KKT conditions corresponding to its problem of maximizing the concave function. The primal-dual variant of the interior-point algorithm relaxes the slackness conditions (2) to the form:

$$\forall i \in \mathcal{N}\ \forall j \in \{1, \ldots, k_i\} \quad \lambda_{ij} h_{ij}(\mathbf{x}_i) = 1/t, \tag{7}$$

where $t > 0$ is a parameter. Feasibility conditions (1) are changed from the inequality to the equality, by introducing the vector of slack variables $\mathbf{s} = [s_{11}, \ldots, s_{ij}, \ldots, s_{Nk_N}]^\intercal$:

$$\forall i \in \mathcal{N}\ \forall j \in \{1, \ldots, k_i\} \quad h_{ij}(\mathbf{x}_i) - s_{ij} = 0. \tag{8}$$

After user selects the accuracy $\epsilon > 0$ and the parameter $\alpha > 0$, the solver starts from a small value of $t = t_0$ and "any" feasible solution $\mathbf{x} = \mathbf{x}_0$. Next, it forms a set of linear equations (1), (3) and (7), by substituting $\mathbf{x}_0$ into them. Based on these equations, the solver computes a Newton step $\Delta\mathbf{x}$, which indicates the direction of maximization. The Newton step is computed from the solution of the system of the following primal-dual equations [23]:

$$\begin{bmatrix} \nabla_x^2 \mathcal{L}_1 & \ldots & \nabla_x^2 \mathcal{L}_N & \mathbf{0} & -\mathbf{H}^\intercal(\mathbf{x}) \\ \mathbf{0} & \ddots & \mathbf{0} & \boldsymbol{\Lambda} & \mathbf{S} \\ h_1(\mathbf{x}_1) & \ldots & h_N(\mathbf{x}_N) & -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_1 \\ \vdots \\ \Delta\mathbf{x}_N \\ \Delta\mathbf{s} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \nabla u_1(\mathbf{x}) - \boldsymbol{\lambda}_1^\intercal h_1(\mathbf{x}_1) \\ \vdots \\ \nabla u_N(\mathbf{x}) - \boldsymbol{\lambda}_N^\intercal h_N(\mathbf{x}_N) \\ \mathbf{S}\boldsymbol{\lambda} - \mathbf{e}/t \\ \mathbf{H}(\mathbf{x}) - \mathbf{s} \end{bmatrix}, \tag{9}$$

where $\mathcal{L}_n = u_n(\mathbf{x}) - \boldsymbol{\lambda}_n^\intercal(h_n(\mathbf{x}_n) - \mathbf{s}_n)$ is the Lagrangian associated with $n$-th player's payoff function, $\boldsymbol{\lambda}$ is the vector of all dual variables, $\mathbf{H}(\mathbf{x})$ is the Jacobian matrix of all constraints $h(\mathbf{x})$, matrix $\boldsymbol{\Lambda}$ is a diagonal matrix of all dual variables, $\mathbf{S}$ is a diagonal matrix of all slack variables, and $\mathbf{I}$ and $\mathbf{e}$ are unit matrix and vector, respectively. The actual step (in both primal $\mathbf{x}$ and dual $\boldsymbol{\lambda}$ variables) is computed using appropriately selected parameters $\boldsymbol{\alpha}$. Here, the solution is updated as follows: $\mathbf{x} \leftarrow \mathbf{x} + \boldsymbol{\alpha}\Delta\mathbf{x}$. If the change in either the solution or the right hand side of (9) is smaller than $\epsilon$, the solver halts.

The algorithm has has been implemented in C++, using BLAS/LAPACK libraries for efficient matrix computations [2]. Note that use of BLAS may allow efficient use of multicore processors. This may be of value when solving large problems. Observe that solving system (9) requires computing Jacobian and Hessian of the system of equations resulting from KKT conditions. This requires calculating derivatives, which, if done numerically, can be moderately time consuming for some functions. In order to alleviate this, the solver allows the user to provide analytically derived expressions for derivatives.

## 6 Experimental study

In the computational experiments we used randomly generated problem instances, defined by the number of players $N$ and the number of charging time intervals $T$ (of constant duration). Table 1 presents the results for $N = 10$ and $T = 5$. For each $n$-th player, energy consumption demands $D_n$ were randomly generated using uniform distribution from the interval $[0, 0.05]$, while battery capacities $C_n$ were randomly generated using uniform distribution from the $[D_n, 0.1]$ interval. Minimal threshold values $L_{n,t}$ were selected from the $[0, 0.01]$ interval, again, using uniform distribution. Function $f$ was $f(x) = 1 - x$.

**Table 1.** Detailed computational results for instance with $N = 10$ and $T = 5$.

| **ID** | $t = 1$ | $t = 2$ | $t = 3$ | $t = 4$ | $t = 5$ | **demand** | **payoff (charge)** |
|---|---|---|---|---|---|---|---|
| 1 | 0.00819 | 0.00916 | 0.01222 | 0.00916 | 0.01127 | 0.022 | 0.045 |
| 2 | 0.00781 | 0.01079 | 0.01279 | 0.01379 | 0.00481 | 0.006 | 0.044 |
| 3 | 0.01107 | 0.00594 | 0.01392 | 0.01092 | 0.00918 | 0.046 | 0.046 |
| 4 | 0.00620 | 0.01524 | 0.00924 | 0.01213 | 0.00720 | 0.035 | 0.045 |
| 5 | 0.00704 | 0.01416 | 0.01516 | 0.00705 | 0.00704 | 0.045 | 0.045 |
| 6 | 0.01112 | 0.00896 | 0.00696 | 0.01086 | 0.01212 | 0.038 | 0.045 |
| 7 | 0.01156 | 0.01252 | 0.01044 | 0.00850 | 0.00754 | 0.046 | 0.046 |
| 8 | 0.00465 | 0.01372 | 0.01172 | 0.01172 | 0.00875 | 0.046 | 0.046 |
| 9 | 0.00899 | 0.00994 | 0.00913 | 0.00908 | 0.01317 | 0.044 | 0.045 |
| 10 | 0.00892 | 0.00678 | 0.01074 | 0.01274 | 0.01082 | 0.015 | 0.044 |

It took the solver 18 iterations to find a Nash equilibrium for $\epsilon = 0.001$, and 6 iterations for $\epsilon = 0.01$. Table 1 contains values of $x_{n,t}$, $D_n$ and $u_n$, for each player from the final iteration. Observe that each player receives approximately the same payoff, which means that the station assigns, in total, approximately the same amount of energy to each vehicle. However, in each time period $t$ the charging rates vary significantly for each car. Overall, all demands are satisfied and allocation is well balanced.

In the second experiment we considered the performance of the proposed algorithm for an increasing number of players $N$. For convenience, we normalized the units of energy to the capacity of the charging station. Hence, the demand of each client was inversely proportional to the total number of clients. Specifically, no car would claim more than $1/2$ unit of energy in each charging interval. Moreover, we assumed that if the total demand in the charging period exceeded one unit, then no charging took place. Thus, functions in equations (6) were:

$$u_n(\mathbf{x}) = \sum_{t=1}^{T} x_{n,t} \left(1 - \sum_{m=1}^{N} x_{m,t}\right).$$

Table 2 compares results of applying our solver with allocations obtained by solving the concave quadratic problem centrally, assuming that all clients' demands are known in advance by a central authority (e.g. they have all been submitted to the charging station that establishes the charging schedule based on its preferences). The first column represents the number of clients $N$. The second column, (**min.sol.**), presents the smallest amount of energy that any player receives in the equilibrium solution, while the column **max.sol.**, states the largest amount of energy that any player receives in the equilibrium solution. In comparison, columns denoted **min.central** and **max.central** contain info about minimal and maximal allocations computed centrally. They have been obtained as a solution to a problem of finding vector $\mathbf{x}$ that maximizes the objective function:

$$U(\mathbf{x}) = \frac{1}{N} \sum_{n=1}^{N} u_n(\mathbf{x}) \tag{10}$$

subject to constraints (4)–(5). This constitutes the average players' payoff, and can be considered as a measure of "social" quality of the solution [13]. These values were obtained using the, state-of-the-art, CPLEX [7] software, which applies the standard barrier interior-point algorithm to solve concave quadratic problems (see, [7] for all details concerning the centralized solution method).

**Table 2.** Comparison of solutions computed for different number of players $N$.

| instance $N$ | **min.sol.** | **max.sol.** | **min.central** | **max.central** | **iterations** | **time** |
|---|---|---|---|---|---|---|
| 10 | 0.061 | 0.13 | 0.021 | 0.15 | 8 | 6 s. |
| 15 | 0.040 | 0.092 | 0.014 | 0.16 | 12 | 39 s. |
| 20 | 0.029 | 0.076 | 0.006 | 0.21 | 9 | 55 s. |
| 25 | 0.015 | 0.061 | 0.004 | 0.21 | 10 | 116 s. |
| 30 | 0.012 | 0.048 | 0.003 | 0.193 | 11 | 216 s. |
| 35 | 0.006 | 0.031 | 0.001 | 0.2 | 11 | 334 s. |
| 40 | 0.002 | 0.027 | 0 | 0.202 | 11 | 492 s. |
| 50 | 0.004 | 0.100 | 0 | 0.221 | 10 | 846 s. |

The solution computed centrally typically allocates large amount of energy to a specific car (e.g., the first client that arrived at the station), while leaving only very little energy for other cars (they are charged just as much as needed to satisfy their minimal demands). This can be seen as an *unfair allocation*. In contrast, the equilibrium solutions (found using the proposed method) tend to balance allocations among clients (differences between the client with the smallest allocated charge and the one with the largest one are relatively small). This can be seen as a *fair allocation* that reduces negative effects of selfishness (due to the threat of loss of payoff that each player takes into consideration). As a result, each client usually receives significantly more than the requested minimum, while no client dominates others in its total allocation. Here, the drawback is that the corresponding values $U(\mathbf{x})$ (average allocated energy) are strictly less than the optimal average values computed centrally. This global performance loss is the price paid for balancing allocations. An interesting question opens here: what is better "unfair optimality" or "fair suboptimality." However, this question cannot be answered on the basis of computational optimization itself.

Moreover, Table 2 lists numbers of iterations needed to reach Nash equilibrium when the requested accuracy was $\epsilon = 0.01$, and the computation time. The number of iterations is almost constant, regardless of the problem size. However, the computational cost of a single iteration raises quickly with increasing $N$, thus computing the charging plans for a large number of cars can become expensive.

## 7 Concluding remarks

In the paper we have demonstrated practical application of theory of non-cooperative concave games to smart energy allocation (charging electric vehicles). The presented approach is distributing power "fairly though suboptimally." In this case, fairness means that the differences between the total amount of allocated energy (to each car) are relatively small. Note that, when human decision-making is considered, fairness is very often considered to be of great value. Moreover, we have described a solver for computing equilibria in non-cooperative convex/concave games with the use of primal-dual interior-point algorithm. We have evaluated its performance for the considered vehicle charging scenario and the results are encouraging.

Across the paper we have indicated a number of directions, which we plan to explore. Some of them are related to the vehicle charging scenario itself, others to the solver. We will report on our findings in subsequent reports.

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