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IDEALIZATION III:

APPROXIMATION AND TRUTH

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ACCURACY, ESSENTIALITY AND IDEALIZATION

1. According to the Idealizational Conception of Science, the scientific enterprise reflects the dialectic nature of reality – in proposing idealizational laws scientists reflect the world's simplicity, by concretizing them they capture the complexity of the world. Most of the time, and perhaps even always, reality does not reveal its simple nature. Those scientists who wish to test their abstract laws in experiments often find that they cannot eliminate all the factors at work. We seem to think intuitively that the more a law is concretized the better the chance that the experimental results will come closer to predictions – the greater accuracy characterizes such a concretization.

Reflecting a little, we realize that there are some phenomena where this is not the case at all. Consider, for example, how (relatively) accurate a calculation of the final velocity of a falling coin can be given that we know its mass and the height from which it falls, for example. In contrast, knowing only two factors is entirely insufficient to allow an even rough estimate whether the coin will fall the head or the tail side up. Those who do not doubt that such a prediction could in principle be made close their eyes to the number of factors that would have to be incorporated in order for the prediction be at least close to the accuracy of predicting the final velocity of a coin fall.

We see thus that some phenomena allow us to predict their outcomes quite accurately on the basis of the approximation of an idealizational law, or some of its close concretizations, and there exist other phenomena where only the approximation of an almost complete concretization can give us comparable accuracy. That this situation is quite intuitive is evidenced by the existence of an opinion – whether prejudiced or not is irrelevant for our purposes – according to which social and perhaps psychological phenomena belong invariably to the second category.

2. How are we to explain this phenomenon on the basis of the idealizational approach to science? Intuitively, it would seem that the notion of essentiality might be involved. According to the Idealizational Conception of Science scientists obey a methodological rule which requires them to concretize an idealizational law in the order reflecting the order of essentiality of factors. The principal factors are incorporated in the idealizational law, the most essential factors are incorporated in the first concretization, the second most essential factors — in the second concretization, etc. It might be supposed that, among other things, the methodological sense of such a rule is that the incorporation of the most essential factors maximizes the accuracy of the idealizational statements proposed first. We shall see below that the intuitive link between the concepts of accuracy and essentiality is not as straightforward as it appears here.

There is a problem, however, in employing the concept of essentiality to conceptualize our puzzle. The concept of essentiality available in the Idealizational Conception of Science is determined for an ordinal scale only. We are thus unable to compare the essentiality of factors of two theories — we can only say what the relative essentiality of factors is within a given theory. For us to be able to begin the kinds of considerations that would lead to a conceptualization of the issue we would have to be able to estimate at least the relative sizes of the essentiality intervals for factors involved in two theories. For this, we need a notion of essentiality defined, at least, on an interval scale. Such a notion (defined, in fact, on the ratio scale) has been made available to us by L. Nowak's [1989a-b] attempt to ground the Idealizational Conception of Science in Unitarian Metaphysics.

3. In [in print] we have shown certain problems with L. Nowak's initial formulation and have proposed to characterize the concepts of influence and essentiality in the following way. We speak of one factor B as being essential to another factor A in virtue of the fact that some value b of factor B influences factor A. A value b of factor B influences factor A if given that B adopts value b it is not the case that A adopts certain values $a_1, ..., a_k$; the set $\{a_1, ..., a_k\}$ is called the exclusion range of A by b of B and symbolized $w(A, B)_b$.

Since an exclusion range is a set we can measure the degree to which a given value of a factor influences another factor by the cardinality of the respective exclusion range. As the concept of essentiality depends on the concept of influence, the following formula defines a suitable measure of the essentiality of factor B for factor A (for details, cf. [in print]):

$$(1) \qquad \mathbf{i}(A, B) = \frac{\sum_{i}^{n_B} |w(A, B)_{b_i}|}{n_B}$$

The essentiality measure is an average of the degree of influence exerted by all values of a given factor on the determined factor.

We can note that this notion of essentiality and influence seems to explain our intuition that essentiality and accuracy are related. The more essential is factor B for A to the greater extent do the values of B restrict and thus determine the values that A can adopt; to put it in a slightly anthropomorphizing way, the more essential is B for A the less freedom does A have in choosing what values to adopt. It appears then that in an extreme case, when B is maximally essential to A, given any value of B there will be only one value that A can adopt.[1] In such a case, a law that incorporates B will afford us maximal accuracy. This is the ideal of determinism.

4. Let us now try to explicate more precisely the concept of accuracy intuitively suggested above. Before we proceed to do so it is advisable to define one more auxiliary concept, maximal influence. For all the considerations below, we suppose that the space of factors essential (called the determining factors) for factor A (called also the determined factor) is composed of one principal factor, B, and m secondary factors, $C_1, \ldots, C_m$. We shall denote the cardinality of the set of all values of A as n_A, of B as n_B, of C_1 as n_{C1}, and so on.

Maximal influence is defined at each level of concretization for those factors whose values exert some influence on the determined factor A. For the k-th concretization, the maximal influence of values of factors $B, C_1, \ldots, C_k$ on A is given by:

$$(2) \qquad n_{Ck} \ldots n_{C1} m_B(n_A - 1)$$

When only the principal factor B is taken into account (for the idealizational law), the maximal influence of values of B is $n_B(n_A-1)$. In view of the restriction mentioned in note 1, a value b of factor B exerts maximal influence on A when b excludes all the values of A except for one, i.e. n_A-1. As B may take n_B values, values of B maximally influence A when each of the values excludes all values of A except for one, hence, the maximal influence of all values of B on A is $n_B(n_A-1)$. When the idealizational law is concretized for the first time, we take into account not only the influence of all the values of B but the influence of all *pairs* of values $<b_i, c_j>$, where b_i is a value of B and c_j is a value of C_j. The maximal influence of values of B and B_1 on A is accordingly $n_{C1}n_B(n_A-1)$.

5. Let us now return to the notion of accuracy. We will assume first that only the principal factor B is taken into account. The absolute accuracy of the idealizational law (incorporating the principal factor B) is defined by the ratio of the total number of values of A excluded by values of B and the maximal influence of values of B on A:

$$(3) \qquad {}^{A}D(B) = \frac{\sum_{i}^{n_B} |w(A, B)_{b_i}|}{n_B(n_A - 1)} = \frac{\mathrm{i}(A, B)}{n_A - 1}$$

We ought to note the close relation between thus defined accuracy and essentiality, which captures the intuition we expressed above quite well. We also see that the absolute accuracy of an idealizational law reaches its maximum value ${}^{A}D(B) = 1$ when B is maximally essential for A, that is, when B fully determines A. Similarly, ${}^{A}D(B)$ has its minimum value (= 0) when B is not essential for A at all.

As the connection between the concept of absolute accuracy of an idealizational law and the measure of essentiality of the principal factor B for A is so close, the former just like the latter averages the influences of the various values of the determining factor. In certain circumstances, a scientist might be interested in extreme (minimal and maximal) values of accuracy. These are readily defined. The maximal absolute accuracy of an idealizational law is a ratio of the cardinality of the exclusion range of that value of B which exerts the greatest influence on A (B_{max}) and the maximal influence of b_{max} on A, viz. $n_A - 1$:

$$(4) \qquad {}^{A}D(B)_{max} = \frac{|w(A, B)_{b_{max}}|}{n_A - 1}$$

The minimal absolute accuracy of the idealizational law ${}^{A}D(B)_{min}$ can be defined analogously. Obviously, ${}^{A}D(B)_{min} = \leqslant {}^{A}D(B) \leqslant {}^{A}D(B)_{max}$.

It can be stipulated that in case of correct conceptualization and good experimental control, the maximal experimental error ought to approximate ${}^{A}D(B)_{min}$, the minimal error and the average error $-$ ${}^{A}D(B)$.

6. Let us now consider the behavior of the concept of absolute accuracy when the law is concretized with respect to the first secondary factor C_1.

Let us first consider a simplified case when B as well as C_1 adopt only one value each: b and c, respectively. The absolute accuracy of the first concretization will be proportional to the number of values of A excluded by values b and c *together*. We are thus concerned with the set-theoretical sum of the influences of b and c, $w(A, B)_b \cup w(A, C_1)_c$, which is the exclusion range of A by the pair $<b, c>$. The absolute accuracy of the first concretization of an idealizational law is defined by the ratio of the cardinality of the exclusion range of A by the pair $<b, c>$ and the maximal influence of that pair, $n_A - 1$. More generally, when we ignore the above simplification, the absolute accuracy of the first concretization of an idealizational law is the ratio of the sum of the cardinalities of all pairs of values of B and C_1, $<b_i, c_j>$, and the maximal influence of the pairs of values of B and C_1 on A, $n_B n_{B1}(n_A - 1)$:

$$(5) \qquad {}^A D(B, C_1) = \frac{\sum_i^{n_B} \sum_j^{n_{C_1}} |w(A, B)_{b_i} \cup w(A, C_1)_{c_j}|}{n_{C_1} n_B (n_A - 1)}$$

The minimal and maximal values of ${}^A D(B, C_1)$ are, as before, 1 and 0, respectively. It should be noticed, however, that in contrast to the concept of absolute accuracy of the idealizational law, the notion of absolute accuracy of its first (in fact, all) concretizations is not so closely related to the concept of essentiality. We shall discuss this relationship below.

We can define the absolute accuracy of the k-th concretization of an idealizational law analogously (we assume as above that B is the principal factor and C_1, ..., C_k, ..., C_m are secondary for A):

$$(6) \qquad {}^A D(B, C_1, ..., C_k) =$$

$$= \frac{\sum_i^{n_B} \sum_j^{n_{C_1}} \cdots \sum_l^{n_{C_k}} |w(A, B)_{b_i} \cup w(A, C_1)_{c_j} \cup ... \cup w(A, C_k)_{c_l}|}{n_{C_k} \ldots n_{C_1} n_B (n_A - 1)}$$

8. We now know how to find out the absolute accuracy of any idealizational statement. A scientist might be interested in knowing also how much the overall (absolute) accuracy is improved by incorporating a particular factor. For this, he will turn to the concept of relative accuracy.

We can define the relative accuracy of the k-th concretization of an idealizational law, or, as we shall also speak, the accuracy afforded by incorporating factor C_k, by the difference between its absolute accuracy and the absolute accuracy of the preceding idealizational statement:

$$(7) \qquad {}^{A}D_w(C_k) = {}^{A}D(B, C_1, ..., C_k) - {}^{A}D(B, C_1, ..., C_{k-1}).$$

The relative accuracy of an idealizational law is identical to its absolute accuracy.

Of course, just as in the case of absolute accuracy we can speak of the maximal and minimal relative accuracy.

8. We might think intuitively that the inclusion of a secondary factor guarantees an increase in the accuracy of a concretized statement incorporating that factor, i.e. that the relative accuracy increases as we carry out subsequent concretizations. This is the case only under very special conditions.

Let us call factors B and C *strictly independent* with respect to A when the exclusion ranges of A by values of B and C do not intersect. We can formulate this condition more precisely thus:

$$(8) \qquad \forall_i \forall_j \; w(A, B)_{b_i} \cap w(A, C)_{c_j} = \emptyset$$

If the factors essential for A, i.e. $B, C_1, ..., C_k$ are strictly independent with respect to A, the absolute accuracy of the k-th concretization is equal to the sum of relative accuracies of those factors (cf. *Appendix*):

$$(9) \qquad {}^{A}D(B, C_1, ..., C_k) = \frac{\mathrm{i}(A, B)}{n_A - 1} + \frac{\mathrm{i}(A, C_1)}{n_A - 1} + ... + \frac{\mathrm{i}(A, C_k)}{n_A - 1} =$$

$$= {}^{A}D_w(B) + {}^{A}D_w(C_1) + ... + {}^{A}D_w(C_k)$$

The assumption of strict independence has two important consequences. First, every concretization leads to an increase of the absolute accuracy: since the essentiality of all factors $B, C_1, ..., C_k$ is strictly greater than 0 and the influences of the values of those factors do not overlap, each factor contributes something to the overall accuracy of the k-th concretization. Secondly, as the relative accuracy allowed for by the incorporation of a factor is proportional to the essentiality of that factor (cf. *Appendix*) the order of the relative accuracies will reflect the essentialist order of the factors, i.e.: $\mathbf{i}(A, B) > \mathbf{i}(A, C_1) > \ldots > \mathbf{i}(A, C_k)$ if and only if ${}^{A}D_w(B) > {}^{A}D_w(C_1) > \ldots > {}^{A}D_w(C_k)$.

It follows that if the relative accuracy of k-th concretization is greater than the relative accuracy of $(k-1)$-th concretization an essentialist error has been made. The rule according to which the order in which the idealizational law is concretized is to reflect the essentialist order of the factors that are thus incorporated has been violated. This shows definitely that the sense of that methodological rule is the maximization of the accuracy of an idealizational law and its close concretizations. The point of this rule is really economical — scientists ought not to waste their energy on the incorporation of factors that produce poor empirical results allowing for relatively low accuracy.

The assumption of strict independence is very strong, however. Let us loosen it a little. We will say that factors B and C are *independent* if the exclusion ranges of A by values of B and C are not included in each other, i.e.:

(10) $\quad \forall_i \forall_j \sim (w(A, B)_{b_i} \subset w(A, C)_{c_j} \vee w(A, C)_{c_j} \subset w(A, B)_{b_i})$

The assumption of (non-strict) independence among factors essential for A is capable of preserving only the first consequence mentioned above. Each concretization step leads to the increase of absolute accuracy but it is not necessarily the case that the more essential a factor the greater the relative accuracy its incorporation affords. No more do we have the guarantee, given by strict independence, that all of the influence of a value of a factor will be significant for the accuracy measure. This time we admit cases where the exclusion range by a value of a factor may overlap with the exclusion range by a value of a previously incorporated factor, in which case theformer's influence is "swallowed" by the influence of the latter. The condition of independence ensures only that each value of a factor that will have some effect on the investigated factor independent of the effect of the previous factors, and so ensures that the absolute accuracy increase — at least minimally.[2]

When this latter assumption is waived, we cannot even guarantee that the incorporation of a factor essential for the investigated one is going to increase the absolute accuracy. If the exclusion ranges of all values of some factor C_k are included in the exclusion ranges of values of the previously incorporated factors $B, C_1, \ldots, C_{k-1}$, the relative accuracy of k-th concretization will be zero. The only guarantee that we have, in this general case, is that the incorporation of a factor is not going to decrease the accuracy afforded by the incorporation of the previous factors.

9. We can now answer the puzzle with which we began. We asked how to conceptualize the well known fact that in the case of some phenomena the incorporation of a few essential factors suffices to obtain pretty accurate predictions whereas in other cases the incorporation of almost all factors is necessary to provide a comparable degree of accuracy. The answer seems simple: the greater the relative accuracy that the incorporation of a given factor affords the fewer factors need to be incorporated to reach a threshold of successful approximation. It ought to be noted that in view of the discussed relation between the essentiality of factors and their relative accuracy, the success of approximation is not related to the essentiality of factors in any simple way *unless* they are strictly independent.

The possibly loose relation between the relative accuracy afforded by factors and their essentiality may prompt us to ask what really determines the construction of an "essentialist" structure of factors – is it the essentiality of a factor or the relative accuracy its incorporation allows for? The measures of essentiality and relative accuracy order factors in the same way only when factors are strictly independent. Short of deciding the "true" ordering relation, it might be supposed that both relations can be used for this purpose. We can accordingly distinguish two attitudes: the attitude of the theoreticians among scientists, who are interested primarily in the essentialist structure of factors, and the attitude of the practitioners of science whose interests are mainly pragmatic and who thus use the accuracy-ordered structure of factors. When factors are strictly independent the theory and the practice go hand in hand.

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APPENDIX

We shall show that if factors essential for A, B, C_1, ..., C_k, are strictly independent with respect to A (condition (8)) the absolute accuracy of the k-th concretization of an idealizational law is equal to the sum of relative accuracies afforded by the factors. We begin by showing that the absolute accuracy of the k-th concretization is proportional to the the sum of the essentiality measures of factors incorporated into the k-th concretization.

We assume that factors B, C_1, ..., C_k are strictly independent with respect to A:

(11) $$\forall_i \forall_j \ldots \forall_l \, (w(A, B)_{b_i} \cap w(A, C_1)_{c_j} \cap \ldots \cap w(A, C_k)_{c_j} = \emptyset$$

Considering (11) we can rewrite (6) as

(12) $$\frac{\sum_{i}^{n_B} \sum_{j}^{n_{C_1}} \ldots \sum_{l}^{n_{C_k}} |w(A, B)_{b_i}| + |w(A, C_1)_{c_j}| + \ldots + w(A, C_k)_{c_l}|}{n_{C_k} \ldots n_{C_1} n_B (n_A - 1)}$$

from which we obtain

(13)

$$\frac{\sum_{i}^{n_B} \sum_{j}^{n_{C_1}} \ldots \sum_{l}^{n_{C_k}} |w(A, B)_{b_i}| + \sum_{i}^{n_B} \sum_{j}^{n_{C_1}} \ldots \sum_{l}^{n_{C_k}} |w(A, C_1)_{c_j}| + \ldots + \sum_{i}^{n_B} \sum_{j}^{n_{C_1}} \ldots \sum_{l}^{n_{C_k}} |w(A, C_k)_{c_l}|}{n_{C_k} \ldots n_{C_1} n_B (n_A - 1)}$$

and this gives us

(14)

$$\frac{n_{C_1} \ldots n_{C_k} \sum_{i}^{n_B} |w(A,B)_{b_i}| + n_b n_{c_2} \ldots n_{C_k} \sum_{j}^{n_{C_1}} |w(A,C_1)_{c_j}| + \ldots + n_B n_{C_1} \ldots n_{C_{k-1}} \sum_{l}^{n_{C_k}} |w(A,C_k)_{c_l}|}{n_{C_k} \ldots n_{C_1} n_B (n_A - 1)}$$

After splitting into fractions we can simplify to

(15) $$\frac{\sum_{i}^{n_B} |w(A, B)_{b_i}|}{n_B (n_A - 1)} + \frac{\sum_{j}^{n_{C_1}} |w(A, C_1)_{c_j}|}{n_{C_1} (n_A - 1)} + \ldots + \frac{\sum_{l}^{n_{C_k}} |w(A, C_k)_{c_l}|}{n_{C_k} (n_A - 1)}$$

Using the definition of essentiality (1) we obtain

$$(16) \qquad \frac{\mathrm{i}(A, B)}{n_A - 1} + \frac{\mathrm{i}(A, C_1)}{n_A - 1} + \ldots + \frac{\mathrm{i}(A, C_k)}{n_A - 1}$$

From the definition of accuracy (7) it follows that

(17)

$$^{A}D_{w'}(C_k) = \left\{ \frac{\mathrm{i}(A, B)}{n_A - 1} + \ldots + \frac{\mathrm{i}(A, C_{k-1})}{n_A - 1} + \frac{\mathrm{i}(A, C_k)}{n_A - 1} \right\} -$$

$$- \left\{ \frac{\mathrm{i}(A, B)}{n_A - 1} + \ldots + \frac{\mathrm{i}(A, C_{k-1})}{n_A - 1} \right\}$$

after subtracting appropriate terms we are left with

$$(18) \qquad ^{A}D_{w'}(C_k) = \frac{\mathrm{i}(A, C_k)}{n_A - 1}$$

from which (9) immediately follows.

NOTES

[1] The exclusion of all values of the determined factor by a value of the determining factor is the maximal influence of that value since we consider here only physical relations between factors (Paprzycka [in print]). The exclusion of all values of the determined factor is a foundation of logical relations between factors.

[2] In fact, already a weaker assumption suffices to reach this conclusion. Let us call factors A and B satisfying the following condition *semi-independent*:

$$(10') \qquad \exists_i \exists_j \sim(w(A, B)_{b_i} \subset w(A, C)_{c_j} \vee w(A, C)_{c_j} \subset w(A, B)_{b_i})$$

Semi-independence guarantees that there be at least one value ofeach factor that will cause the increase of the overall accuracy. It will be noticed at the same time that an analogical weakeningof condition (8) (replacing the universal by existential quantifiers) although obviously ensures that the accuracy increasewith every concretization it does not secure the stronger consequence, viz. that the increase in accuracy is proportional tothe essentiality of an incorporated factor.

REFERENCES

Nowak, L. [1989a]. Byt i Myśl. Przyczynek do metafizyki unitarnej (Being and Thinking. A Contribution to the Unitarian Metaphysics). *Studia Filozoficzne*, **258**, 5-18.

Nowak, L. [1989b]. On the (Idealizational) Structure of Economic Theories. *Erkenntnis* **30**, 225-246.

Paprzycka, K [in print]. Metafizyczne dziury. Przyczynek do koncepcji zależności w metafizyce unitarnej (Metaphysical Holes. A Contribution to the Conception of Dependence in the Unitary Metaphysics). To appear in *Poznańskie Studia z Filozofii Nauki*.

Paprzycka, K. and M. Paprzycki [in print]. Kilka uwag o istotności i procedurach deformacyjnych (Several Remarks on Essentiality and Deformational Procedures). To appear in *Poznańskie Studia z Filozofii Nauki*.