ROBUST TIME-OPTIMAL POSITIONAL STABILIZATION CONTAINING A SPEED LIMITATION TASK

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Abstract. The present paper deals with the time-optimal positional control for objects whose dynamics are described by a random differential inclusion with discontinuous right-hand side. Considerations include the speed limitation problem. The existence and characteristics of the almost certain time-optimal control, proved in recent works, are commented upon. In compliance with the results obtained, the switching curve, well-known from the (deterministic) classical case, has been "blurred" by the random factor introduced here to the switching area. Empirical examinations confirm numerous advantages of the control systems designed, especially in the area of robustness.

Keywords. Mathematical system theory; minimum time control; position control; discontinuous differential inclusion; stochastic control; random processes; control system design; suboptimal control; robustness

Résumé. Le présent article étudie le contrôle de position optimal en temps des objets dont la dynamique est décrite par l'inclusion différentielle aléatoire avec le côté droit discontinu. Ces considérations incluent la question de la limitation de la rapidité. L'existence et la caractéristique du contrôle optimal en temps presque sûr, démontrés dans des travaux récents, sont commentées. En accord avec les résultats obtenus la courbe de commutation, bien connue, du cas classique (déterministe), a été "délavée" par l'introduction du facteur aléatoire à l'aire des commutations. Les expériences empiriques ont confirmés plusieurs avantages du système de contrôle projeté, spécialement dans le cadre de la robustesse.

1. INTRODUCTION

The dynamics of a broad class of time-optimal controlled objects are described by the following differential inclusion, given below in the operator form:

$$\ddot{y} \in H + u \;, \tag{1}$$

where u is a bounded control function, y denotes the position of the object, and the function H represents a model of motion resistances. If one omits this factor, i.e. when $H \equiv 0$, formula (1) naturally expresses the second law of Newtonian mechanics for straight-line or rotary motion. In many practical problems the speed $\dot{y}$ has to be limited.

The simplest example of those objects consists of various industrial automata and robots, reversing mills as well as many other plants operating mainly through a change of position in particular mechanisms. Such devices are called positional.

The time-optimal stabilization of system (1), or the task of reaching the equilibrium state $\dot{y}(t) = y(t) = 0$ in a minimal and finite time, will be considered in this paper.

The essential element of model (1) is the multivalued (set-valued) function H describing motion resistances. For the majority of practically appearing types of these resistances, that function can be expressed in the following form:

$$H(\dot{y}(t), y(t), t) = v(\dot{y}(t), y(t), t)\, F(\dot{y}(t)) \;, \tag{2}$$

where v denotes a real bounded continuous function, and F is a real bounded piecewise continuous function, additionally multivalued in the points of discontinuities. (For example, suppose for the sake of illustration the simplest form of this function

$$F(\dot{y}(t)) = SGN(\dot{y}(t)) = \\ = \begin{cases} 1 & if \;\; \dot{y}(t) > 0 \\ [-s,s] & if \;\; \dot{y}(t) = 0 \\ -1 & if \;\; \dot{y}(t) < 0 \;\; , \end{cases} \qquad (3)$$

where the parameter $s > 1$ is connected with static friction.)

In many technical problems, the form of the applied model of motion resistances has great influence on the complexity or even the feasibility of a successful analysis. In the model adopted here, it is assumed that the function v introduced in equation (2) is the realization of a given stochastic process V with almost all the realizations being continuous and jointly bounded. Therefore, the dependence of the function v on $\dot{y}(t)$, $y(t)$ and t, is replaced by the dependence on a random factor. Moreover, such a model also regards as probabilistic uncertainty the dependence of motion resistances on a number of other factors, not only $\dot{y}(t)$, $y(t)$ and t, but also those which are usually omitted in the deterministic approach, due to the necessity to simplify the model. The probabilistic concept proposed here also considers perturbations and noise occurring in the system.

The present paper constitutes a continuation of the work described in (Kulczycki, 1992, 1993) additionally supplemented with the task of the speed limitation (Kulczycki, 1995).

2. MAIN RESULTS

First, some notions used in the following will be made more precise.

Let T be an interval with nonempty interior, $t_0 \in T$, and $\mathfrak{R}$ denote the real space.

Consider the deterministic differential inclusion

$$\dot{x}(t) \in g(x(t),t) \qquad (4)$$

with the initial condition

$$x(t_0) = x_0 \;\; , \qquad (5)$$

where $g : \mathfrak{R}^n \times T \to \mathcal{P}(\mathfrak{R}^n)$, $\mathcal{P}(A)$ denotes the set of subsets of A, and $x_0 \in \mathfrak{R}^n$. The function $x : T \to \mathfrak{R}^n$ absolutely continuous on every compact subinterval of the set T, is a solution of differential inclusion (4):

- in the Caratheodory sense (C-solution), if it fulfills inclusion (4) almost everywhere in T,
- in the Filippov sense (F-solution), if

$$\dot{x}(t) \in F[g](x(t),t) \quad \text{almost everywhere in } T \;\; , \qquad (6)$$

where the operator F is defined as

$$F[g](x(t),t) = \bigcap_{\substack{Z \subset \mathfrak{R}^n: \\ m(Z)=0}} conv\big[g\big((x(t)+eB)\setminus Z, t\big)\big] \;\; , \qquad (7)$$

B denotes the open unit ball in the space $\mathfrak{R}^n$, m is the Lebesgue measure, and $conv[D]$ means the convex closed hull of the set D. The C- or F-solutions of deterministic differential inclusion (4) with initial condition (5) are unique, if all C- or F-solutions, respectively, are identically equal functions.

Let (Ω, Σ, P) be a probability space.

Suppose the random differential inclusion

$$\dot{X}(\omega,t) \in G(\omega, X(\omega,t), t) \;\; , \qquad (8)$$

with the initial condition

$$X(\omega,t_0) = X_0(\omega) \quad \text{for almost all } \omega \in \Omega \;\; , \qquad (9)$$

where $G : \Omega \times \mathfrak{R}^n \times T \to \mathcal{P}(\mathfrak{R}^n)$ and X_0 is an n-dimensional random variable defined on (Ω, Σ, P). An n-dimensional stochastic process X defined on (Ω, Σ, P) and T, is an almost certain C- or F-solution of random differential inclusion (8), if almost all its realizations are C- or F-solutions, respectively, of the related deterministic differential inclusions received at the fixed $\omega \in \Omega$. Almost certain C- or F-solutions of random differential inclusion (8) with initial condition (9) are unique, if all almost certain C- or F-solutions, respectively, are equivalent stochastic processes (i.e. $P(\{ \omega \in \Omega : \bar{X}(\omega,t) = \bar{\bar{X}}(\omega,t) \}) = 1$ for every $t \in T$).

Finally, let $U_a \subset \{u : T \to \Re^m\}$ denote a set of admissible controls. Consider the function $G : \Omega \times \Re^n \times \Re^m \times T \to \mathcal{P}(\Re^n)$ which defines the dynamics of the random system submitted to the control U

$$\dot{X}(\omega,t) \in G(\omega, X(\omega,t), U(\omega,t), t) \ , \qquad (10)$$

and initial condition (9). Then, the m-dimensional stochastic process U^o defined on (Ω, Σ, P) and T, will be called an almost certain time-optimal control, if almost all its realizations are time-optimal controls for proper deterministic systems obtained from dependencies (9)-(10) at the fixed $\omega \in \Omega$.

The theorem proved by Kulczycki (1995), whose thesis constitutes the solution of the time-optimal positional stabilization problem considered here, will be presented now.

Theorem

Let:

(a) $t_0 \in \Re$, $T = [t_0, \infty)$, $w > 0$, $x_0 \in \Re \times [-w, w]$, and (Ω, Σ, P) be a complete probability space;

(b) origin of coordinates constitute a target set;

(c) $U_a = \{u : T \to [-1,1]\}$ represent a set of admissible controls, and $X_a = \Re \times [-w, w]$ a set of admissible states;

(d) V mean a real stochastic process defined on (Ω, Σ, P) and T, with almost all realizations being continuous and fulfilling the boundary condition $V(\omega,t) \in [v_-, v_+]$ for $t \in T$, where $\varnothing \neq [v_-, v_+] \subset (-1,1)$;

(e) $f : \Re \to [-1,1]$ denote a piecewise continuous function fulfilling locally a Lipschitz condition except at discontinuity points and $z \cdot f(z) \geq 0$ for every $z \in \Re$, and also $F : \Re \to \mathcal{P}([-1,1])$ be such that

$$F(z) = \begin{cases} f(z) & if \quad z \neq z_i \\ F_i & if \quad z = z_i \end{cases} \ , \qquad (11)$$

where F_i means a subset of the interval $[-1,1]$ and z_i is any real number, for $i = 1, 2, \ldots, k$;

(f) the random differential inclusion describing the dynamics of the system submitted to the control U

$$\dot{X}_1(\omega,t) = X_2(\omega,t) \qquad (12)$$

$$\dot{X}_2(\omega,t) \in U(\omega,t) - V(\omega,t)\, F(X_2(\omega,t)) \qquad (13)$$

with the initial condition

$$\begin{bmatrix} X_1(\omega,t_0) \\ X_2(\omega,t_0) \end{bmatrix} = x_0 \quad \text{for almost all } \omega \in \Omega \ , \qquad (14)$$

be given.

Then:

(A) there exists an almost certain time-optimal control U^o;

(B) realizations of this control take on the following forms:

$$U^o(\omega,t) = \begin{cases} -1 & for \ \ t \in [t_0, t_1(\omega)) \\ V(\omega,t)\, F(-w) & for \ \ t \in [t_1(\omega), t_2(\omega)) \\ +1 & for \ \ t \in [t_2,(\omega), \infty) \end{cases} \qquad (15)$$

or

$$U^o(\omega,t) = \begin{cases} +1 & for \ \ t \in [t_0, t_1(\omega)) \\ V(\omega,t)\, F(w) & for \ \ t \in [t_1(\omega), t_2(\omega)) \\ -1 & for \ \ t \in [t_2,(\omega), \infty) \end{cases} \ , \qquad (16)$$

where $t_0 \leq t_1(\omega) \leq t_2(\omega) < \infty$ for every $\omega \in \Omega$;

(C) the above control generates a unique almost certain C-solution;

(D) the above solution is also a unique almost certain F-solution;

(E) the values of almost all realizations of these solutions belong to the set X_a for every $t \in T$. ■

In the proof of the above Theorem (Kulczycki, 1995), the set of admissible states X_a has been divided into the following disjoint subsets: Q_+, Q_-, R_+, R_-, and the origin being a target (Fig. 1). First, let K_{+-}, K_{++}, denote sets of all states from X_a which can be brought to the origin by the control $u \equiv +1$, if $v \equiv v_-$ or $v \equiv v_+$, respectively; analogically K_{--} and K_{-+} for $u \equiv -1$, if $v \equiv v_-$ or $v \equiv v_+$, respectively. Next, let the following sets be given:

$$Q_+ = \{[x_1,x_2]^T \in X_a \text{ such that there exist } [x_1',x_2]^T \in K_{++} \text{ and } [x_1'',x_2]^T \in K_{+-} \text{ with } x_1' \le x_1 \le x_1''\} \tag{17}$$

$$Q_- = \{[x_1,x_2]^T \in X_a \text{ such that there exist } [x_1',x_2]^T \in K_{--} \text{ and } [x_1'',x_2]^T \in K_{-+} \text{ with } x_1' \le x_1 \le x_1''\} \tag{18}$$

$$R_+ = \{[x_1,x_2]^T \in X_a \backslash Q \text{ such that there exists } [x_1',x_2]^T \in Q \text{ with } x_1 < x_1'\} \tag{19}$$

$$R_- = \{[x_1,x_2]^T \in X_a \backslash Q \text{ such that there exists } [x_1',x_2]^T \in Q \text{ with } x_1' < x_1\} , \tag{20}$$

where $Q = Q_+ \cup \{[0,0]^T\} \cup Q_-$.

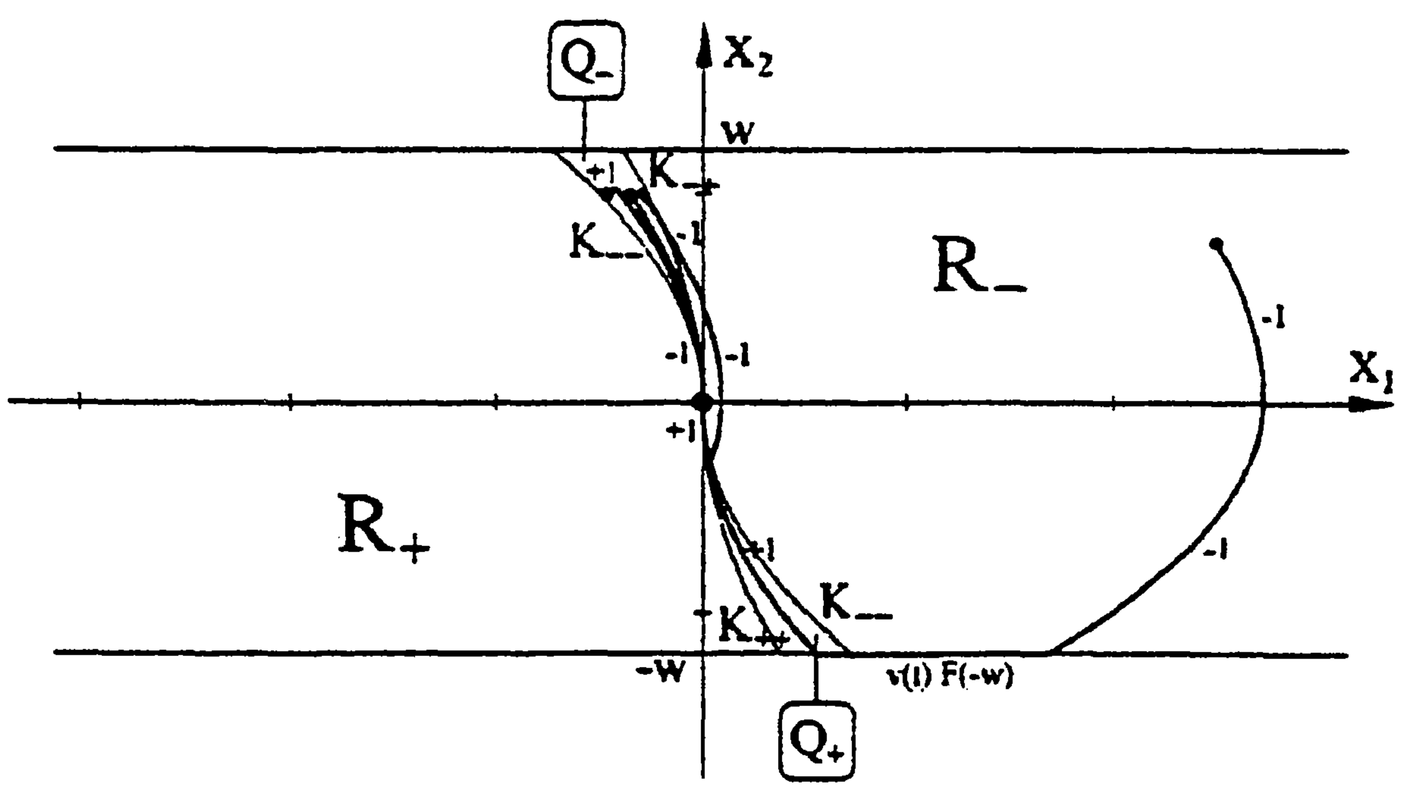

Fig. 1. Almost certain time-optimal control

Let now $\omega \in \Omega$, or thereby $V(\omega,\cdot) \equiv v$, be fixed. If $x_0 \in R_-$ (Fig. 1), then there exist t_1 and t_2 such that the solution generated by the control

$$u^o(t) = \begin{cases} -1 & for \ t \in [t_0,t_1) \\ v(t)\,F(-w) & for \ t \in [t_1,t_2) \\ +1 & for \ t \in [t_2,\infty) \end{cases} \tag{21}$$

reaches the target in a minimum time t_f, with $t_0 < t_1 \le t_2 < t_f < \infty$, $x(t_2) \in Q_+$, and if $t_1 \ne t_2$ then also $x(t) \in \mathcal{R} \times \{-w\}$ for $t \in [t_1,t_2]$. Analogically, if $x_0 \in R_+$, then there exist t_1 and t_2 such that the solution generated by the control

$$u^o(t) = \begin{cases} +1 & for \ t \in [t_0,t_1) \\ v(t)\,F(w) & for \ t \in [t_1,t_2) \\ -1 & for \ t \in [t_2,\infty) \end{cases} \tag{22}$$

reaches the target in a minimum time t_f, with $t_0 < t_1 \le t_2 < t_f < \infty$, $x(t_2) \in Q_-$, and if $t_1 \ne t_2$ then also $x(t) \in \mathcal{R} \times \{w\}$ for $t \in [t_1,t_2]$. Let $x_0 \in Q_-$ (Fig. 1); for particular $\omega \in \Omega$ the time-optimal control can be of the form (21) or (22), with additional conditions as above, or the following:

$$u^o(t) = -1 \quad for \quad t \in [t_0,\infty) \ . \tag{23}$$

The case $x_0 \in Q_+$ is similar; the counterpart of control (23) is

$$u^o(t) = +1 \quad for \quad t \in [t_0,\infty) \ . \tag{24}$$

The proof of the time-optimality of the above described control functions u^o is based on the theory of differential inequalities.

The function $U^o : \Omega \times T \to [-1,+1]$ defined by the formula $U^o(\omega,\cdot) \equiv u^o$, where u^o was assigned above at fixed $\omega \in \Omega$, is a stochastic process and the required almost certain time-optimal control fulfilling the conditions formulated in the thesis of the Theorem presented.

In assumption (d) of this Theorem there appears the stochastic process V with almost all realizations being continuous and jointly bounded. The condition of boundedness can be examined by one-dimensional distributions of that stochastic process; however, the Kolmogorov theorem formulates the sufficient condition for the continuity of almost all realizations of a stochastic process on the basis of properties of two-dimensional distributions. Finally, the requirements formulated above with respect to the stochastic process V are thus mutually independent and identifiable on the basis of finite dimensional distributions (Wong, 1971; Chapter II).

The relation $[v_-, v_+] \subset (-1,1)$ ensures the controllability of the system. The assumption that the function F fulfills Lipschitz condition has been introduced to guarantee the uniqueness and equality of C- and F-solutions.

The condition $z \cdot F(z) \geq 0$ included in assumption (e) has been formulated only for the sake of clarity of notation. At any rate, this inequality is physically justified, because with positive values of the stochastic process V it is consistent with the property of the energy dissipation.

3. CONCLUSIONS AND APPLICATIONS

The switching curve γ, well-known from the classical case of time-optimal transfer of a mass (Athans and Falb, 1966; Section 7.2), has been generalized in the above Theorem to a switching area Q ($\gamma = Q$ if $v_- = v_+ = 0$ and $w = \infty$). Namely, the function H, defined in the Introduction and representing the model of motion resistances, has been decomposed into two factors: $F(\dot{y}(t))$ and $V(\omega,t)$. The former, deterministic one, having only insignificant influence on the complexity of the theoretical analysis, made it possible to incorporate the properties of discontinuity and multivalence of friction phenomena. The latter one, thanks to its probabilistic nature, includes among other things approximations and identification errors (of the first factor too), motion resistances dependence on position, time and temperature, as well as perturbations and noise naturally occurring in real systems. The switching curve (even more general than γ, i.e. only with the condition $v_- = v_+ \in (-1,1)$) which is implied by the first, deterministic factor, has been "blurred" by the second, random one to the switching area.

An almost certain time-optimal control ensures the realization of the minimum of the expected value of the time for reaching a target set; however, it is additionally dependent on the random factor, in practice unknown *a priori*. The result of this dependence is that the above control is difficult to apply directly, but it constitutes a useful basis for the creation of technical constructions of suboptimal structures in which the direct dependency of the control function on the random factor is eliminated.

Namely, according to the results of the Theorem presented, the suboptimal feedback controller function, independent of a random factor, can be defined by the following formula:

$$U^s(X(\omega,t)) = \begin{cases} 1 & \text{if } X(\omega,t) \in R_+ \cup \mathfrak{R} \times (-\infty, -w) \\ d_1 & \text{if } X(\omega,t) \in R_+ \cap \mathfrak{R} \times [w-e, w] \\ d_2 & \text{if } X(\omega,t) \in Q_+ \\ 0 & \text{if } X(\omega,t) \in \{[0,0]^T\} \\ -d_2 & \text{if } X(\omega,t) \in Q_- \\ -d_1 & \text{if } X(\omega,t) \in R_- \cap \mathfrak{R} \times [-w, -w+e] \\ -1 & \text{if } X(\omega,t) \in R_- \cup \mathfrak{R} \times (w, \infty) \end{cases} , \quad (25)$$

where $-1 < d_1 < 1$, $0 < d_2 \leq 1$, and $0 < e < w$ (Fig. 2). In practice these values can be obtained heuristically. Particularly, the parameter d_1 would be approximately equal to the mean of motion resistances; however, the parameter d_2 should be close to 1. Only almost certain F-solutions occur in the system thus obtained; C-solutions do not exist in general case. These F-solutions are unique.

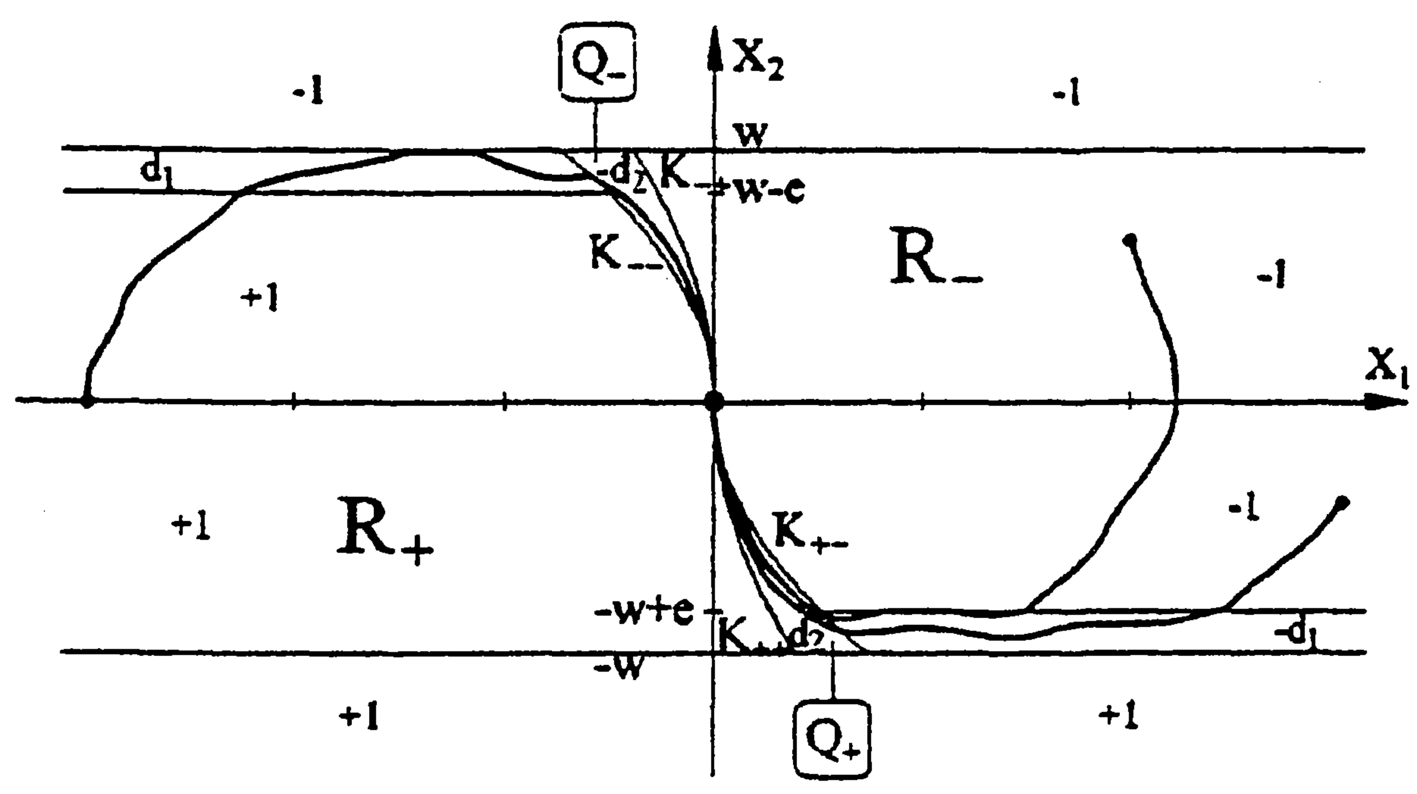

Fig. 2. Suboptimal control with constant parameters d_1 and d_2

The parameters d_1 and d_2 can be constant as in the above concept, but they can also vary, increasing continuously from the value $1 - v_+ + v_-$ on the set K_{+-} up to the value 1 on the set K_{++}, from -1 on K_{-+} to $-1 + v_+ - v_-$ on K_{--}, and from -1 on the sets

$R_+ \cap \Re \times \{w\}$, $R_- \cap \Re \times \{-w+e\}$ to the value 1 on $R_+ \cap \Re \times \{w-e\}$, $R_- \cap \Re \times \{-w\}$, respectively (Fig. 3). This makes it possible to achieve a result similar to the bicycle-racing track or bob-sleigh track, which are horizontal on the interior part, and become more vertical the farther they go to the outside. The value of the parameter d_2 should be equal to 1 even in the neighborhood of the sets K_{++} and K_{-+}, with the aim of neutralizing the most unfavorable realizations of the stochastic process V. Almost certain F-solutions occur in a system thus designed. Moreover, the rules for variations of the parameters d_1 and d_2 values postulated above ensure also the existence of almost certain C-solutions, equal to those F-solutions. Both types of the above solutions are unique.

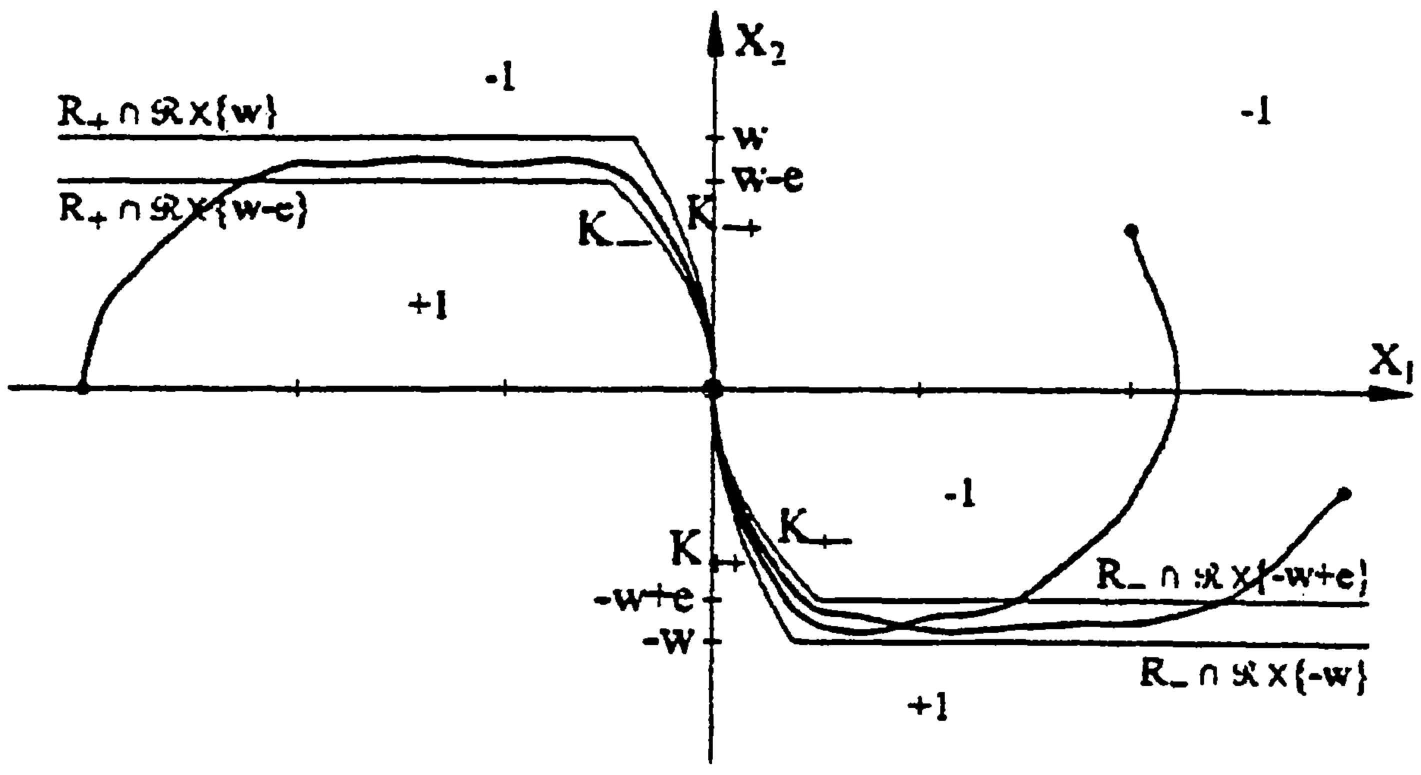

Fig. 3. Suboptimal control with inconstant parameters d_1 and d_2

If the constraints of an actuator limit the control to the extreme values of the admissible set, the results of the Theorem may be modified according to the physical observation that the influence of motion resistances in both periods of time - before and after a switching - can be averaged. Thus, after performing a detailed analysis of the sensitivity of the control system to the values of motion resistances, one can use elements of statistical decision theory, where a loss function is connected with extending the time for reaching the target if the control switching has been too late or too early. A detailed description of such a control structure will be the subject of a separate publication. In the general case there are no C-solutions in the system thus obtained; almost certain F-solutions exist, and they are unique.

The probabilistic concept of the control systems designed in the present paper have been successfully empirically verified (Kulczycki, 1995). It should be underlined that the control system constructed turned out to be only slightly sensitive to the inaccuracy resulting from identification and perturbations - robustness is a very valuable property of random control systems.

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