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Time-Optimal Positional Feedback Controller for Random Systems

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ABSTRACT

This paper discusses the problem of a time-optimal positional control with the use of a random, discontinuous and multivalued (set-valued) model of motion resistances. The control structure, defined as a closed-loop system with a deterministic feedback controller function, is investigated on the basis of statistical decision theory. Empirical examinations have confirmed the correct operation of the control system, and have demonstrated its numerous advantages, especially in the area of robustness. In order to verify the formal correctness of the control design, the existence and some features of a solution of a random discontinuous differential inclusion describing the dynamics of the obtained system are shown. The presented paper also contains many suggestions for practical applications.

Key Words: time-optimal control, positional control, random object, discontinuous differential inclusion, robustness, decision theory

I. Introduction

The dynamics of a broad class of time-optimal controlled objects is described by the following differential inclusion, given below in the operator form:

$$\ddot{y} \in H + u\,, \tag{1}$$

where u is a bounded control function, y denotes the position of the object, and the function H represents a model of motion resistances. If one omits this factor, i.e. when $H \equiv 0$, formula (1) expresses the second law of Newtonian mechanics for straight-line or rotary motion.

The most representative example of such systems is a great number of industrial plants, called positional, which operate by changing the positions of particular mechanisms, e.g. machine attachments, reversing mills, and especially automata and robots (Tourassis, 1988). Control that yields the minimum of time for these changes becomes, obviously, a very important economic aim.

The essential element of model (1) constitutes a bounded multivalued (set-valued) function H describing the dependence of motion resistances on many factors, such as the position and velocity of the object, temperature, etc. The form of that function is of primary influence on the complexity of the time-optimal control problem. Thus, this complexity makes deterministic analysis impossible without significant simplification concerning the function H, which in turn limits the feasibility of application.

In this paper a probabilistic concept is proposed. The following random model of motion resistances, inspired by their physical properties, will be applied here:

$$H(\dot{y}(t), \omega) = v(\omega) F(\dot{y}(t))\,, \tag{2}$$

where v denotes a real random variable, and F is a piecewise continuous function additionally multivalued at the points of discontinuity. The function F describes the complicated dependence of friction phenomena on velocity, which by nature is discontinuous and multivalued. (For example, let this function, for the sake of illustration, be expressed in the simplest form:

$$F(\dot{y}(t)) = \mathrm{SGN}(\dot{y}(t)) = \begin{cases} 1 & \text{if } \dot{y}(t) > 0 \\ [-s, s] & \text{if } \dot{y}(t) = 0 \\ -1 & \text{if } \dot{y}(t) < 0 \end{cases}, \tag{3}$$

where parameter $s>1$ is connected with static friction.) However, the random variable v regards as a probabilistic uncertainty the dependence of motion resistances on a number of other factors, e.g. position or temperature, which are usually omitted in the deterministic approach due to the necessity of simplifying the model. Moreover, such a concept also considers the perturbations occurring in the system.

Because this random variable can also have a one-point distribution, such a probabilistic problem represents a generalization of the basic task of a time-optimal positional control with a discontinuous model of motion resistances (Hejmo, 1987). In the special case, where $H \equiv 0$, this deterministic task is reduced to the classical case of the time-optimal transfer of a mass (Athans and Falb, 1966).

This paper presents an applicational conclusion for the concept described in Kulczycki (1993).

Finally, the following task will be considered Let:

(1) $t_0 \in \mathbb{R}$, $T=[t_0, \infty)$;

(2) $x_0=[x_{01}, x_{02}]^T \in \mathbb{R}^2$ and $x_T=[x_{T1}, x_{T2}]^T \in \mathbb{R}^2$ represent initial and target states, respectively;

(3) f: $\mathbb{R} \to [-1, 1]$ denote a piecewise continuous function with discontinuity points $z_1, z_2, \ldots, z_k$, and F: $\mathbb{R} \to \mathcal{P}([-1, 1])$ be such that

$$F(z) = \begin{cases} f(z) & \text{if } z \neq z_i \\ F_i & \text{if } z = z_i \end{cases}, \tag{4}$$

where $\mathcal{P}(A)$ means the set of subsets of the set A, and $i=1, 2, \ldots, k$;

(4) (Ω, Σ, P) be a complete probability space; (From a practical point of view, the assumption of the completeness does not reduce the generality of the following considerations (Rudin, 1974a).)

(5) v denote a real random variable defined on the space (Ω, Σ, P), such that $P(\{\omega \in \Omega: v(\omega) \in (-1, 1)\})=1$;

(6) a random differential inclusion

$$\dot{X}_1(\omega, t)=X_2(\omega, t) \tag{5}$$

$$\dot{X}_2(\omega, t) \in U(\omega, t)-v(\omega)F(X(\omega, t)) \tag{6}$$

with an initial condition

$$\begin{bmatrix} X_1(\omega, t_0) \\ X_2(\omega, t_0) \end{bmatrix} = x_0 \quad \text{for almost every } \omega \in \Omega \tag{7}$$

describe the dynamics of the positional system submitted to a control U.

For the above random system, the problem of a time-optimal feedback controller is considered, where a function U_R: $\mathbb{R}^2 \to [-1, 1]$ such that

$$U(\omega, t)=U_R(X_1(\omega, t), X_2(\omega, t)), \tag{8}$$

due to practical demands does not directly depend on the random factor.

Because of the discontinuity of the function F, the theory of generalized solutions of differential equations (Hajek, 1979) will be used. This theory is extended over differential inclusions and the random case in the following. Mathematical background and notions will be used in accordance with Rudin (1974a, 1974b) and Wong (1971).

First, a deterministic differential inclusion

$$\dot{x}(t) \in g(x(t), t) \tag{9}$$

with an initial condition

$$x(t_0)=x_0, \tag{10}$$

where g: $\mathbb{R}^n \times T \to \mathcal{P}(\mathbb{R}^n)$, T denotes an interval with nonempty interior and $t_0 \in T$, $x_0 \in \mathbb{R}^n$, will be considered.

A function x: $T \to \mathbb{R}^n$, absolutely continuous on every compact subinterval of the set T and fulfilling condition (10), is a solution of differential inclusion (9)-(10):

- in the Caratheodory sense (C-solution), if it fulfills inclusion (9) almost everywhere in T,
- in the Filippov sense (F-solution), if:

$$\dot{x}(t) \in \mathcal{F}[g](x(t), t) \text{ almost everywhere in } T, \tag{11}$$

where

$$\mathcal{F}[g](x(t), \mathrm{t}) = \bigcap_{e>0} \bigcap_{Z \subset \mathbb{R}^n : m(Z)=0} \mathrm{conv}[g((x(t)+eB) \backslash Z, t)], \tag{12}$$

B denotes the open unit ball in the space $\mathbb{R}^n$, m is the Lebesgue measure, and $\mathrm{conv}[L]$ means the convex closed hull of the set L.

A C- or F-solution of the deterministic differential inclusion (9)-(10) is unique, if all C- or F-solutions, respectively, are identically equal functions.

In general, a C-solution does not have to be an F-solution and vice versa. Nevertheless, most frequently, an F-solution constitutes a considerable generalization of a C-solution (Hajek, 1979).

The idea of a C-solution represents in practice a mathematical formalization of "joining" classical solutions (Hubbard and West, 1991), applied in the case of function g discontinuities with respect to t. If this function is discontinuous also with respect to $x(t)$, then a C-solution often does not exist, which implies the necessity of considering an F-solution. In particular, an F-solution properly describes the so-called sliding trajectory (Slotine and Li, 1991), well known from engineering practice.

The above concepts of solutions will be extended in what follows to cover the random case. Thus, a random differential inclusion

$$\dot{X}(\omega,\ t)\in G(\omega,\ X(\omega,\ t),\ t) \tag{13}$$

with an initial condition

$$X(\omega,\ t_0)=X_0(\omega)\quad \text{for almost all } \omega\in\Omega, \tag{14}$$

where G: $\Omega\times\mathbb{R}^n\times T\to\mathcal{P}(\mathbb{R}^n)$ and X_0 is an n-dimensional random variable defined on (Ω, Σ, P), will be considered now.

An n-dimensional stochastic process X defined on (Ω, Σ, P) and T, is an almost certain C- or F-solution of the random differential inclusion (13)-(14), if almost all its realizations are C- or F-solutions, respectively, of the proper deterministic differential inclusions received at a fixed $\omega\in\Omega$.

An almost certain C- or F-solution of the random differential inclusion (13)-(14) is unique, if all almost certain C- or F-solutions, respectively, are equivalent stochastic processes, i.e. $P(\{\omega\in\Omega: X^{\sim}(\omega,t)=X^{=}(\omega,t)\})=1$ for every $t\in T$.

II. Feedback Controller for a Deterministic Task

In the following section, an auxiliary deterministic task will be considered. Let the random factor ω, therefore, the value of the random variable v, be fixed. This value will hereafter be denoted by w, i.e. $v(\omega)=w$.

Suppose that x_+ and x_- are unique C-solutions of the system with the condition $x(0)=x_T$, defined on the interval $(-\infty, 0]$, and generated by the control $u\equiv+1$ or $u\equiv-1$, respectively. Also, let:

$$K_+=\{x_+(t) \text{ for } t<0\} \tag{15}$$

$$K_-=\{x_-(t) \text{ for } t<0\} \tag{16}$$

$$R_+=\{\begin{bmatrix}x_1\\x_2\end{bmatrix}\in\mathbb{R}^2 \text{ such that there exists} \begin{bmatrix}x_1^*\\x_2^*\end{bmatrix}\in K \text{, with } x_1>x_1^* \text{ and } x_2=x_2^*\} \tag{17}$$

$$R_-=\{\begin{bmatrix}x_1\\x_2\end{bmatrix}\in\mathbb{R}^2 \text{ such that there exists} \begin{bmatrix}x_1^*\\x_2^*\end{bmatrix}\in K \text{, with } x_1>x_1^* \text{ and } x_2=x_2^*\}, \tag{18}$$

where $K=K_-\cup\{x_T\}\cup K_+$. The time-optimal control is expressed by the following formula (Hejmo, 1987):

$$u(t)=u_r(x_1(t),\ x_2(t))$$

$$=\begin{cases}-1 & \text{if } [x_1(t), x_2(t)]^T\in(R_-\cup K_-)\\ 0 & \text{if } [x_1(t), x_2(t)]^T=x_T\\ +1 & \text{if } [x_1(t), x_2(t)]^T\in(R_+\cup K_+)\end{cases} \tag{19}$$

and the set K constitutes a switching curve (Fig. 1, 2).

In the time-optimal feedback controller equations, i.e. formulas (15)-(19), parameter w intervenes because it influences the form of the trajectories x_+, x_- and, therefore, the shape of the switching curve K. However, in the random system, this value will be unknown a priori. The analysis of the system sensitivity to the value of parameter w, which is presented in the following, will then be of great importance. Thus, the value of parameter w occurring in the object is still denoted as w; however, the value assumed in feedback controller equations will be marked by W.

The case where the second coordinate of the target state is equal to zero, i.e. with $x_{T2}=0$, will be presented first.

If $W=w$, the control is time-optimal (Fig. 1). The state of the system is brought to the switching curve by a C-solution, and, being permanently included in this curve hereafter, it also reaches the target along a C-solution.

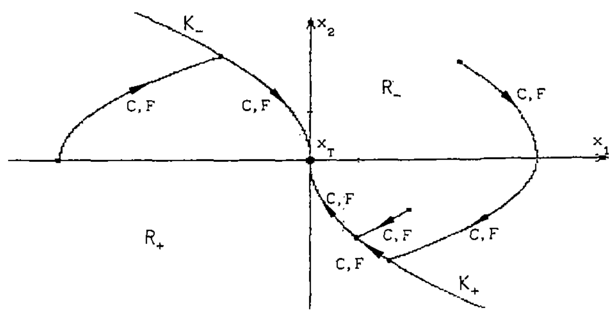

Fig. 1. Deterministic feedback controller and representative trajectories in the case $x_{T2}=0$.

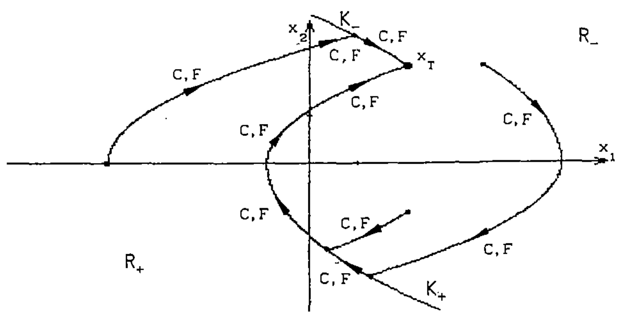

Fig. 2. Deterministic feedback controller and representative trajectories in the case $x_{T2}\neq0$.

The trajectory representative for the case $W>w$ is shown in Fig. 3. As a result of the fact that it has oscillations around the target, over-regulations occur in the system. The state reaches the target in a finite time along a C-solution.

Figure 4 shows trajectories representative for the case $W<w$. Until the switching curve is crossed, C-solutions occur, followed only by F-solutions: sliding trajectories appear in the system. The target is reached in a finite time.

In both of the last two cases, i.e. with $W\neq w$, the time needed to reach the target increases from the optimal one approximately proportionally to the difference between the values W and w.

The remaining case, $x_{T2}\neq 0$, will be presented now. Let in particular $x_{T2}>0$; investigations for $x_{T2}<0$ can be made analogically.

If $W=w$ (Fig. 2), the considerations are identical to those before for $x_{T2}=0$.

In the case $W>w$ (Fig. 5), C-solutions occurring in the system create a limit cycle: the target is not reached.

Finally, in the case $W<w$ (Fig. 6), only a part of the trajectories (marked on Fig. 6 with arrows) reach the target in a finite time; until the switching curve is crossed, C-solutions occur, followed only by F-solutions (sliding trajectories appear). Other trajectories reach only the end point x_e placed at the crossing of

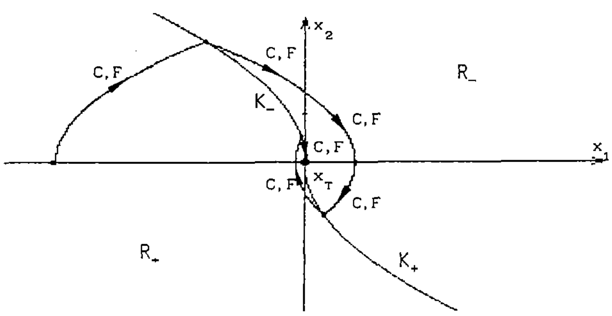

Fig. 3. Trajectory representative for $W>w$ in the case $x_{T2}=0$.

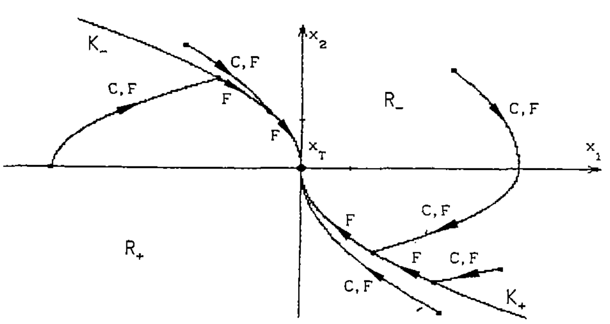

Fig. 4. Trajectories representative for $W<w$ in the case $x_{T2}=0$.

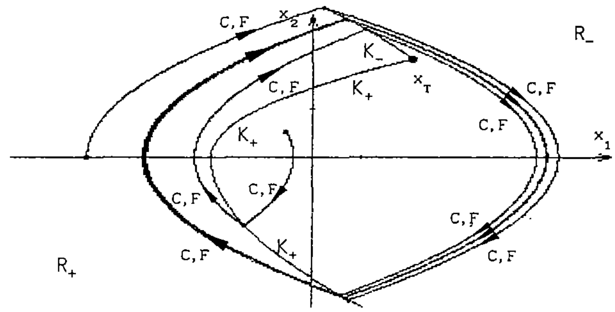

Fig. 5. Trajectories representative for $W>w$ in the case $x_{T2}\neq 0$.

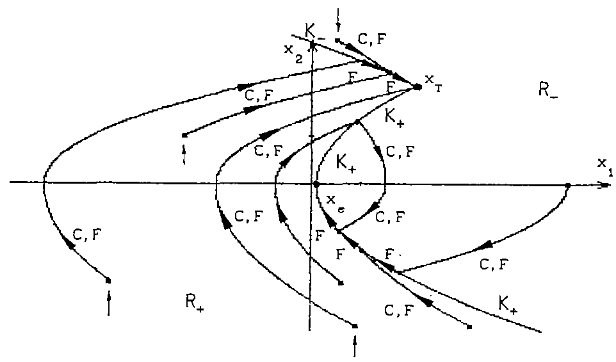

Fig. 6. Trajectories representative for $W<w$ in the case $x_{T2}\neq 0$.

the axis x_1 and the switching curve-the state does not reach the target then; until the switching curve is crossed by a trajectory in the lower half-plane of the state space, C-solutions occur, followed only by F-solutions (sliding trajectories).

All the phenomena above were shown by Hejmo (1987). The fact that C-solutions are also F-solutions results from Hajek (1979) with the additional definition $F(z_i)=\lim_{z\to z_i^+} f(z)$ for $i=1, 2, ..., k$. This artificial condition is acceptable for the control system described above because before the target is reached, the condition $x_2(t)\in\{z_1, z_2, ..., z_k\}$ is fulfilled only on a zero-measure subset of the set T.

If $x_{T2}\neq 0$ and $W\geq w$, nonunique solutions also exist in the system. Then, as the time needed to reach the target, the following value $t_u = \inf_x t_{u,x}$ is assumed, where $t_{u,x}$ denotes the time for reach the target by a solution x generated by a control u. The trajectories selected in such a way have the proper technical interpretation in the problem considered.

III. Feedback Controller for a Random Problem

In this section, the random positional system (5)-(7), which is the subject of the present paper, will be investigated. The value of parameter w which appears in the deterministic feedback controller equations (15)-(19) happens to be a random variable in the random system considered here. The value of the qloove variable is naturally unknown a priori with probability 1. It should, therefore, due to formula (8), be determined in a statistical way. With this aim, some elements of statistical decision theory (Berger, 1980) will be used.

The basic task of this theory is the optimal selection of one element from all possible decisions only on the basis of probabilistic information about the state of nature, especially without knowledge of its real state.

Let the following be given: a non-empty set N of possible states of nature, a non-empty set D of possible decisions, and a function

$$l:\ D\times N \rightarrow \mathbb{R}\cup\{\pm\infty\} \tag{20}$$

representing losses. Denote by l_m: $D\rightarrow \mathbb{R}\cup\{\pm\infty\}$ the minimax loss function

$$l_m(d) = \sup_{n\in N} l(d,n)\,. \tag{21}$$

If, additionally, on the set N, a probability space (N, S, P) is defined, and for every $d\in D$ the integral $\int_N l(d,n)\,dP(n)$ exists, suppose that l_b: $D\rightarrow \mathbb{R}\cup\{\pm\infty\}$ is the Bayes loss function defined as

$$l_b(d) = \int_N l(d,n)\ dP(n)\,. \tag{22}$$

Every element $d_m\in D$ such that

$$l_m(d_m) = \inf_{d\in D} l_m(d) \tag{23}$$

is called a minimax decision, and analogically, every element $d_b\in D$ such that

$$l_b(d_b) = \inf_{d\in D} l_b(d) \tag{24}$$

is called a Bayes decision. The above procedures for obtaining these elements are said to be minimax and Bayes rules, respectively.

The main difference between those decision reles is their interpretation. This results directly from the form of the loss functions l_m and l_b: the "pessimistic" minimax rule assumes the occurrence of the most unfavorable state of nature and opposes it while the "realistic" Bayes rule allows the action to be taken which is best in the sense of expectation.

In the problem of a time-optimal control investigated in this paper, the value $W\in D\subset(-1, 1)$ assumed in the feedback controller equations will be treated as a decision while the value $\nu(\omega)=w\in N\subset(-1, 1)$ of the random variable occurring in the system will be considered as the unknown state of nature. The loss function is defined for $(W, w)\in D\times N$, and its values are related to the time needed to reach the target, if in the feedback controller equations the value W was assumed, but in the object the value w occurs.

Again, the case $x_{T2}=0$ will be investiqated first. The following suggestions for the determination of the value of parameter W result from analysis of the deterministic system presented in the previous section.

If over-regulations can be allowed, it is worthwhile to use the Bayes rule with real values of the loss function. Such a choice is possible because the determination of the value of parameter W such that it is either smaller, equal, or greater than w allows the system state to be brought to the target in a finite time. (However, this time increases approximately proportionally to the difference between the values W and w.)

If over-regulations are not allowed, this determination should be carried out on the basis of the minimax rule, assuming infinite values of the loss function for $W>w$. This makes it possible to avoid over-regulations because they occur only if $W>w$.

Now let $x_{T2}\neq 0$. The case $x_{T2}>0$ will be considered; investigations for $x_{T2}<0$ are analogical.

The case $W=w$ is impossible to obtain in practice. However, the determination of the value of parameter W that is either greater or smaller than w precludes reaching the target from any initial state because of the occurrence of the cycle (Fig. 5) or existence of the end point (Fig. 6). In the proposed feedback controller, the switching curve K will be divided into three parts. The division points will be the target and the crossing point with the axis x_1. For every part, there will be differently determined values of parameter W, which for particular parts are defined in the following as W_1, W_2 and W_3.

The value of parameter W_1defining the part of the switching curve K_-, or K for $x_2\in[x_{T2}, \infty)$ (see also Fig. 8), should be determined using the minimax rule with infinite values of the loss function for $W>w$. This choice is made in order to avoid generation of a limit cycle, which appears when the value of parameter W_1 is greater than w. If, however, this value is smaller than w, the state of the system is brought to the target in a finite time.

For the determination of the value of parameter W_2 defining the part K_+ for $x_2\in[0, x_{T2}]$, it is necessary to apply the minimax rule with infinite values of the loss function for $W<w$. This is because an overly large value of parameter W_2 allows the state to be brought to the part defined by parameter W_1, which, as was demonstrated above, can be successfully determined.

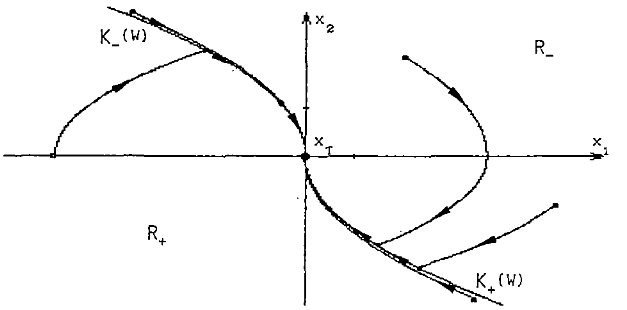

Fig. 7. Random feedback controller obtained in Example III.2, and representative trajectories for the case x_{T2}=0.

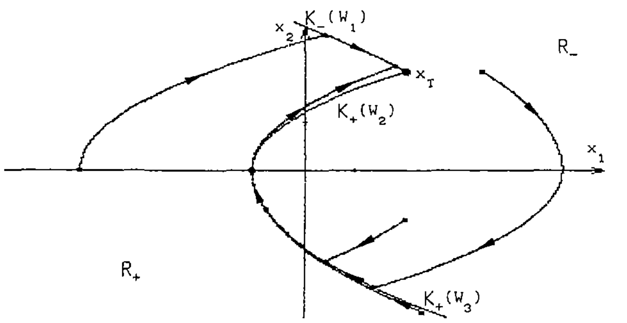

Fig. 8. Random feedback controller obtained in Example III.2, and representative trajectories for the case $x_{T2}\neq 0$.

A overly small one, however, causes the occurrence of the end point, whose existence is not admissible from the point of view of utility.

Finally, the value of parameter W_3 defining the part K_+ for $x_2 \in (-\infty, 0]$ can be obtained using the Bayes rule with real values of the loss function. Both overly small and large values of this parameter are acceptable because this makes it possible for the state to be brought to the parts defined by parameters W_1 and W_2, which can be successfully determined as shown above.

To summarize, if the switching curve is constructed according to the above principles, then the control defined by the formula

$$U(\omega, t)=U_R(X_1(\omega, t), X_2(\omega, t))$$

$$=\begin{cases} -1 & \text{if } [X_1(\omega,t), X_2(\omega,t)]^T \in (R_- \cup K_-) \\ 0 & \text{if } [X_1(\omega,t), X_2(\omega,t)]^T = x_T \\ +1 & \text{if } [X_1(\omega,t), X_2(\omega,t)]^T \in (R_+ \cup K_+) \end{cases} \tag{25}$$

brings the state of the system to the target along an F-solution, with probability 1.

A vital problem concerning differential equations and inclusions with a discontinuous right-hand side is the existence of their solutions. The theorem stated below solves this task.

Theorem III.1. Let t_0, T, x_0, x_T, f, F, (Ω, Σ, P) and v fulfill assumptions (1)-(5) formulated in the Introduction.

If the sets K_-, K_+, R_- and R_+ are obtained according to the principles formulated in Section III, then the control defined by formula (25) generates an almost certain F-solution of the random differential inclusion (5)-(6) with initial condition (7).

The lemmas used to prove this theorem are included in Appendix 1 whereas the theorem itself is shown in Appendix 2.

It is worthwhile to notice that, except in specific cases, an almost certain C-solution does not exist while the above almost certain F-solution is not unique.

Example III.2. Suppose that v is a random variable whose distribution has a density function h, with support of the form $[w_*, w^*] \subset (-1, 1)$; moreover, let this function be continuous and positive in (w_*, w^*). The loss function will be described by the following formula:

$$l(W, w) = \begin{cases} -p(W-w) & \text{if } W-w \leq 0 \\ q(W-w) & \text{if } W-w \geq 0 \end{cases}, \tag{26}$$

where $p, q \in \mathbb{R} \cup \{\infty\}$, with only one of them being infinite. In the case of an infinite value, let $\infty \cdot 0 = 0$.

According to the above assumptions, it is accepted that $N=D=[w_*, w^*]$.

With the fixed value of parameter W, from the definitions of the minimax and Bayes loss functions (21) and (22), the following are the results, respectively:

$$l_m(W)=\max(\{-p(W-w^*), q(W-w_*)\}) \tag{27}$$

$$l_b(W) = \int_{w_*}^{W} q(W-w)\,h(w)\,dw - \int_{W}^{w^*} p(W-w)\,h(w)\,dw. \tag{28}$$

If $p=\infty$, then from dependence (27) it can be obtained that the infimum of the function l_m on the set D is realized by

$$W=w^*, \tag{29}$$

and if $q=\infty$, then this infimum is assumed for

$$W=w_*. \tag{30}$$

The values indicated by formulas (29) and (30) constitute the desired minimax decision with infinite values of the loss function for $W<w$ and $W>w$, respectively.

However, with real positive values p and q, the function l_b is differentiable in the interval(w_*, w^*); therefore, one obtains:

$$l_b'(W) = p \int_{w^*}^{W} h(w)\,dw + q \int_{w_*}^{W} h(w)\,dw \qquad (31)$$

and analogously

$$l_b''(W) = (p+q)\,h(w)\,. \qquad (32)$$

Using formula (31), the equivalence of the following conditions can be proved by elementary transformations:

$$l_b'(W) = 0 \qquad (33)$$

$$\frac{p}{q} = \frac{\int_{w_*}^{W} h(w)\,dw}{\int_{W}^{w^*} h(w)\,dw} \qquad (34)$$

$$\int_{w_*}^{W} h(w)\,dw = \frac{p}{p+q}\,. \qquad (35)$$

Formula (32) implies that the function l_b'' is positive in the set (w_*, w^*); therefore, the function l_b here is strictly convex. Because $0<\frac{p}{p+q}<1$, dependence (35), equivalent to condition (33), is fulfilled only in one point, so in this point the function l_b assumes its minimum, which is global in the set$D=[w_*, w^*]$ thanks to the continuity of this function in the points w_* and w^*.

The value W fulfilling conditions (34) and equivalently (35) constitutes the desired Bayes decision with real values of the loss function. It ex presses a so-called quantile of order $\frac{p}{p+q}$ (Fisz, 1963). Condition (34) is easily interpreted. Namely, the value of parameter W is chosen in such a way that the quotient of the probabillities that the random variable v will take on values smaller and greater than it, equals $\frac{p}{q}$, which is compatible with intuition considering the form of loss function (26). Moreover, this value does not depend straightforwardly on parameters p and q, but only on their ratio. However, condition (35) is more useful for calculation.

The assumptions concerning the random variable v formulated in this example are fulfilled by all practically applied types of nondiscrete distributions with a bounded support, including uniform, triangular and beta (Fisz, 1963). The beta distribution approximates reality in the best way, but the uniform one is the easiest to use. In particular, for the uniform distribution, criterion (35) takes on the form of a simple algebraic equation while in the case of the triangular distribution, it proves to be a conditional algebraic equation. For the beta distribution, an integrand function occurring in formula (35) has the form of a so-called binomial differential, which does not have a primitive function except in special cases (Fichtenholz, 1979); thus, the use of this criterion here requires numerical procedures. In practice, those distributions can be considered complementary and should be applied according to the amount of reliable data possessed and the time available for calculation. Another method of solving criterion (35) based on the artificial neural networks technique has been investigated by Kulczycki and Schioler (1994).

To summarize, in accordance with the considerations stated before, if the values of parameters W or W_1, W_2, W_3 should be determined due to the minimax rule with infinite values of the loss function for $W<w$ or $W>w$, or the Bayes rule with real values of this function, then they can be obtained from formulas (29), (30) and (35), respectively. In all three cases, those values are determined uniquely.

If one possesses the obtained values W or W_1, W_2, W_3, the feedback controller equations can be calculated. Thus, for the exemplary function F described by formula (3), the equations of the switching curve K take on the form

$$x_1 = \frac{x_2^2 - x_{T2}^2}{2(-1-W_1)} + x_{T1} \quad \text{for } x_2 \in [x_{T2}, \infty) \qquad (36)$$

$$x_1 = \frac{x_2^2 - x_{T2}^2}{2(1-W_2)} + x_{T1} \quad \text{for } x_2 \in [0, x_{T2}] \qquad (37)$$

$$x_1 = \frac{x_2^2}{2(1+W_3)} - \frac{x_{T2}^2}{2(1-W_2)} + x_{T1} \quad \text{for } x_2 \in (-\infty, 0] \qquad (38)$$

in the case $x_{T2}>0$. (Formula (36) defines the set K_- while formulas (37) and (38) point out the set K_+.) For $x_{T2}<0$, the equations are analogical. If $x_{T2}=0$, one should substitute into dependences (36) and (38): $W=W_1=W_3$ (formula Eq. (37) has no meaning here). The sets R_- and R_+ constitute adequate areas resulting from division of the plane $\mathbb{R}^2$ by the curve K according to formulas (17) and (18). For the sets K_-, K_+, R_-, R_+ obtained above, the value of the control is defined by dependence(25). Figures 7 and 8 provide an illustration of the above results.

IV. Conclusions

In this paper a probabilistic way of solving the problem of a time-optimal positional control has been presented. Theoretical considerations have led to the design of a closed-loop control system, according to practical requirements, with a deterministic feedback controller function. A large number of completed empirical examinations (Kulczycki, 1992) have confirmed the theoretical considerations and proved the correct operation of the system. The target state was reached in every case with the assumed precision of the scale 0.1-0.5% of the initial state norm. If it was assumed that over-regulations were unacceptable, they did not occur during the control process. Only a slight dependence of feedback controller parameters on the initial and final conditions in previous recourses, which affected the data received, was recorded. However, with different characteristics (e.g. changes in time) of motion resistances, the differences could be considerable. In practice, it is thus necessary to revise the data; therefore, an adaptive structure is suggested. The constructed control system turns out to be slightly sensitive to the inaccuracy resulting from identification and to perturbations; this is a very valuable property of random control systems.

Appendix 1. (lemmas)

Lemma 1.1. Let $I\subset\mathbb{R}$ be a non-empty, measurable and bounded set. Then, the space $L^\infty(I,\mathbb{R}^n)$ is a Borel subset of the space $L^1(I,\mathbb{R}^n)$.

Proof. Because for $h\in L^1(I,\mathbb{R}^n)$ and any fixed $a<b\in\mathbb{Q}\cap I$, the mapping $h\mapsto\int_a^b\|h(y)\|\,dy$ is clearly continuous, then for every $k\in\mathbb{N}$, the set

$$D_{a,b,k}=\{h\in L^1(I,\mathbb{R}^n):\int_a^b\|h(y)\|\,dy\le k\,(b-a)\}\tag{39}$$

is closed in the space $L^1(I,\mathbb{R}^n)$. Therefore, directly from the definition of a Borel set, it is enough to prove:

$$L^\infty(I,\mathbb{R}^n)=\bigcup_{k\in\mathbb{N}}\bigcap_{a<b\in\mathbb{Q}\cap I}D_{a,b,k}.\tag{40}$$

The inclusion "$\subset$" is obvious.

The proof of the inclusion "$\supset$" will be carried out by contradiction. Assume the existence of the function $h\in L^1(I,\mathbb{R}^n)\setminus L^\infty(I,\mathbb{R}^n)$ and $k^*\in\mathbb{N}$ such that $h\in D_{a,b,k^*}$ for all $a<b\in\mathbb{Q}\cap I$.

Because $h\notin L^\infty(I,\mathbb{R}^n)$, then especially for k^*, there exists a positive Lebesgue measure set $Z\subset I$ such that $\|h(y)\|>k^*$ if $y\in Z$, or

$$\int_Z\|h(y)\|\,dy>k^*\,m(Z).\tag{41}$$

From the characteristics of the sets measurable in the Lebesgue sense, the result is that a set S of the type F_σ satisfying $S\subset Z$ and $m(Z\setminus S)=0$ exists; therefore

$$\int_S\|h(y)\|\,dy>k^*\,m(S).\tag{42}$$

The set S is of the type F_σ; thus denoting its components as A_i, and defining $A_i^{\sim}=A_1\cup A_2\cup\ldots\cup A_i$ (i=1, 2, ...) one obtains the increasing family of the sets $A_i^{\sim}$, where

$$S=\bigcup_{i=1}^{\infty}A_i^{\sim}.\tag{43}$$

Thanks to inequality (42) and the fact that $h\in L^1(I,\mathbb{R}^n)$, due to the Lebesgue Dominated Convergence Theorem, a number $i^*\in\mathbb{N}\setminus\{0\}$ exists such that

$$\int_{A_{i^*}^{\sim}}\|h(y)\|\,dy>k^*\,m(S);\tag{44}$$

therefore,

$$\int_{\mathrm{int}(A_{i^*}^{\sim})}\|h(y)\|\,dy>k^*\,m(S).\tag{45}$$

Moreover, the set $\mathrm{int}(A_{i^*}^{\sim})$ is open; thus there exists an increasing family B_j (j=1, 2, ...) of closed intervals with rational ends (and their finite unions, in the case where $\mathrm{int}(A_{i^*}^{\sim})$ is disconnected), such that the set $\mathrm{int}(A_{i^*}^{\sim})$ is their union. As before, there exists a number $j^*\in\mathbb{N}\setminus\{0\}$ such that

$$\int_{B_{j^*}}\|h(y)\|\,dy>k^*\,m(S).\tag{46}$$

Because $m(B_{j^*})\le m(S)$, formula (46) contradicts $h\in D_{a,b,k^*}$ for all $a<b\in\mathbb{Q}\cap I$. Equality (40) is, therefore, shown, which ends the proof of Lemma 1.1. ■

Lemma 1.2. Let $I\subset\mathbb{R}$ be a non-empty and measurable set. Then, the space $L^\infty_{loc}(I,\mathbb{R}^n)$ is a Borel subset of the space $L^1_{loc}(I,\mathbb{R}^n)$.

Proof. Let any $y\in I$ be fixed, and denote $I_m=I\cap[y-m,\,y+m]$ for every $m\in\mathbb{N}\setminus\{0\}$. From the properties of the topology of the space $L^1_{loc}(I,\mathbb{R}^n)$, it directly results that for any $m\in\mathbb{N}\setminus\{0\}$, the restriction mapping r_m: $L^1_{loc}(I,\mathbb{R}^n)\to L^1(I_m,\mathbb{R}^n)$ is continuous. Therefore, it is Borel. Because Lemma 1.1 yields that $L^\infty(I_m,\mathbb{R}^n)$ is a Borel subset of the space $L^1(I_m,\mathbb{R}^n)$, the inverse image $r_m^{-1}(L^\infty(I_m,\mathbb{R}^n))$ is a Borel subset of $L^1_{loc}(I,\mathbb{R}^n)$.

$L^\infty_{loc}(I,\mathbb{R}^n)$ is clearly a subset of the space $L^1_{loc}(I,\mathbb{R}^n)$, so the following is obviously true:

$$L^\infty_{loc}(I,\mathbb{R}^n)=\{h\in L^1_{loc}(I,\mathbb{R}^n):h|_{I_m}\in L^\infty(I_m,\mathbb{R}^n)\}$$
$$=\bigcap_{m=1}^{\infty}r_m^{-1}(L^\infty(I_m,\mathbb{R}^n)).\tag{47}$$

This implies that the space $L^\infty_{loc}(I,\mathbb{R}^n)$, as the countable intersection of Borel sets, is a Borel subset of the space $L^1_{loc}(I,\mathbb{R}^n)$, which ends the proof of Lemma 1.2. ■

Appendix 2. (proof of Theorem III.1)

Let the set Ω be restricted to those ω which satisfy the condition $v(\omega)\in(-1,\,1)$. The remaining subset of Ω has zero measure and can thus be omitted due to the assumed completeness of the probability space and the definition of an almost certain F-solution.

This proof will consist of seven steps leading to the construction of the almost certain F-solution sought.

(1) First considered will be the set

$$M_w=\{[x_1,\ x_2,\ y_1,\ y_2]^T\in\mathbb{R}^4:\ [y_1,\ y_2]^T\in\mathcal{F}[g]([x_1,\ x_2]^T,\ t)\}, \quad \textbf{(48)}$$

where $\mathcal{F}$ is given by formula (12), and g denotes the right-hand side of the random differential inclusion (5)-(6), (25), parameterized by the random variable value $v(\omega)=w$.

Equation (5) yields $y_1=x_2$; therefore, according to formulas (6) and (25), this set takes on the following form:

$$M_w=\bigcup_{j=1}^{2k+1}(\{[x_1,\ x_2,\ x_2,\ y_2]^T:\ x_2\in H_j,\ [x_1,\ x_2]^T\in R_+,$$

$$y_2\in +1-wF(x_2)\}\cup\{[x_1,\ x_2,\ x_2,\ y_2]^T: x_2\in H_j,\ [x_1,\ x_2]^T\in K,$$

$$y_2\in[-1,\ +1]-wF(x_2)\}\cup\{[x_1,\ x_2,\ x_2,\ y_2]^T:\ x_2\in H_j,$$

$$[x_1,x_2]^T\in R_-,\ y_2\in -1-wF(x_2)\}), \quad \textbf{(49)}$$

where, for z_i being the points of discontinuities of the function F, the sets H_j are of the form $(-\infty,\ z_1)$, $\{z_1\}$, $(z_1,\ z_2)$, $\{z_2\}$, ..., $(z_k,\ \infty)$. The set M_w constitutes, therefore, the union of a finite number of submanifolds, the parameterization of which depends continuously on w, that is, the function c_d: $(-1,\ 1)\times\mathbb{R}^4\to\mathbb{R}$ defined by

$$c_d(w,\ [x_1,\ x_2,\ y_1,\ y_2]^T)=\mathrm{dist}(M_w,\ [x_1,\ x_2,\ y_1,\ y_2]^T), \quad \textbf{(50)}$$

where dist denotes the distance of a set from a point, is continuous.

(2) Consider for every fixed $t\in T$ a mapping c_t: $(-1,\ 1)\times L^\infty_{loc}(T,\mathbb{R}^2)\to\mathbb{R}$ such that

$$c_t(w,y)=c_d\Big(w,x_{01}+\int_{t_0}^{t}y_1(z)\,dz,\ x_{02}+\int_{t_0}^{t}y_2(z)\,dz,$$

$$\limsup_{s\to 0^+}\frac{1}{m(T_{t,s})}\int_{T_{t,s}}y_1(t+z)\,dz,$$

$$\limsup_{s\to 0^+}\frac{1}{m(T_{t,s})}\int_{T_{t,s}}y_2(t+z)\,dz\Big), \quad \textbf{(51)}$$

where $y=[y_1,\ y_2]^T$, $T_{t,s}=T\cap[t-s,\ t+s]$, and m denotes the Lebesgue measure.

From the properties of the topology of the space $L^\infty_{loc}(T,\mathbb{R})$, the continuity of the mappings

$$y_1\mapsto x_{01}+\int_{t_0}^{t}y_1(z)\,dz \quad \textbf{(52)}$$

$$y_2\mapsto x_{02}+\int_{t_0}^{t}y_2(z)\,dz \quad \textbf{(53)}$$

$$y_1\mapsto\limsup_{s\to 0^+}\frac{1}{m(T_{t,s})}\int_{T_{t,s}}y_1(t+z)\,dz \quad \textbf{(54)}$$

$$y_2\mapsto\limsup_{s\to 0^+}\frac{1}{m(T_{t,s})}\int_{T_{t,s}}y_2(t+z)\,dz \quad \textbf{(55)}$$

clearly results. Therefore, the mapping c_t, which is the composition of the continuous mappings (52)-(55) and c_d, is also continuous.

(3) Now, define a mapping c_s: $(-1,\ 1)\times L^\infty_{loc}(T,\mathbb{R}^2)\to\mathbb{R}\cup\{\infty\}$ by

$$c_s(w,y)=\sup_{t\in T}c_t(w,y). \quad \textbf{(56)}$$

As an upper envelope of the continuous mappings c_t, the mapping c_s is lower semicontinuous; therefore, it is Borel.

(4) Finally, suppose a mapping c: $\Omega\times L^\infty_{loc}(T,\mathbb{R}^2)\to\mathbb{R}\cup\{\infty\}$ such that

$$c(\omega,\ y)=c_s(v(\omega),\ y). \quad \textbf{(57)}$$

Being is the composition of the measurable mapping v with the Borel mapping c_s, it is measurable. Therefore, the set $c^{-1}(0)$ is measurable in $\Omega\times L^\infty_{loc}(T,\mathbb{R}^2)$. Moreover, thanks to Lemma 1.2 proved in Appendix 1, the inclusion mapping $L^\infty_{loc}(T,\mathbb{R}^2)$ into $L^1_{loc}(T,\mathbb{R}^2)$ is Borel, so $c^{-1}(0)$ is measurable also as a subset of $\Omega\times L^1_{loc}(T,\mathbb{R}^2)$.

(5) Following the definitions of the succeeding mappings introduced in points (1)-(4), it is readily shown that the set $c^{-1}(0)$ constitutes also a graph of the multivalued mapping E: $\Omega\to\mathcal{P}(L^1_{loc}(T,\mathbb{R}^2))$ as $E(\omega)$ is a set of the elements $y\in L^1_{loc}(T,\mathbb{R}^2)$ such that a function x: $T\to\mathbb{R}^2$ defined by the formula

$$x(t)=\Big[x_{01}+\int_{t_0}^{t}y_1(z)\,dz,\ x_{02}+\int_{t_0}^{t}y_2(z)\,dz\Big]^T \quad \textbf{(58)}$$

is an F-solution of the random differential inclusion (5)-(7), (25), parameterized by the value of the random variable v on $\omega\in\Omega$.

The facts described in Section III imply that such a solution exists for every $\omega\in\Omega$; therefore, as a continuous function, it belongs to the space $L^1_{loc}(T,\mathbb{R}^2)$; thus $E(\omega)\neq\phi$ for every $\omega\in\Omega$. From (Sainte-Beuve, 1974), the existence of a measurable selector of the mapping E, or the measurable mapping e: $\Omega\to L^1_{loc}(T,\mathbb{R}^2)$ such that $e(\omega)\in E(\omega)$, results directly. (The assumptions of the Sqinte-Beure theorem are clearly fulfilled because the space $L^1_{loc}(T,\mathbb{R}^2)$ is separable and complete metric.)

(6) Consider a function X: $\Omega\times T\to\mathbb{R}^2$ defined as

$$X(\omega,\ t)=x_{\omega,e}(t)\ \text{for}\ \omega\in\Omega,\ t\in T, \quad \textbf{(59)}$$

where $x_{\omega,e}$ denotes the function x expressed by formula (58) for ω and chosen by the selector e. Now, it will be shown that X is a stochastic process.

Let $t\in T$ and $m\in\mathbb{N}\setminus\{0\}$ fulfilling $t<m$ be fixed. The function $X(\bullet,\ t)$ is a composition of the following mappings:

$$e:\Omega\to L^1_{loc}(T,\mathbb{R}^2) \qquad e(\omega)=y \quad \textbf{(60)}$$

$$e_1:L^1_{loc}(T,\mathbb{R}^2)\to L^1(T_m,\mathbb{R}^2) \qquad e_1(y)=y|_{T_m} \quad \textbf{(61)}$$

$$e_2:L^1(T_m,\mathbb{R}^2)\to C(T_m,\mathbb{R}^2) \qquad e_2(y|_{T_m})=x|_{T_m} \quad \textbf{(62)}$$

$$e_3:C(T_m,\mathbb{R}^2)\to\mathbb{R}^2 \qquad e_3(x|_{T_m})=x|_{T_m}(t), \quad \textbf{(63)}$$

where e, x, y are defined in point (5) of this proof, and

$T_m = T \cap [-m, m]$. The mapping e is measurable according to its definition. The properties of the topology of the space $L^1_{loc}(T, \mathbb{R}^2)$ imply straightforwardly the continuity of the mapping e_1. From the inequality

$$\| e_2[y^{\sim}](s) - e_2[y^{\approx}](s) \| = \| \int_{t_0}^{s} (y^{\sim}(z) - y^{\approx}(z))\, dz \|$$

$$\leq \int_{t_0}^{s} \| y^{\sim}(z) - y^{\approx}(z) \| dz \leq \int_{T_m} \| y^{\sim}(z) - y^{\approx}(z) \| dz, \quad \textbf{(64)}$$

true for every $s \in T_m$ and $y^{\sim}, y^{\approx} \in L^1(T_m, \mathbb{R}^2)$, the continuity of the mapping e_2 results. The continuity of the mapping e_3 is obvious.

The above properties imply the measurability of the function $X(\bullet, t)$ for any $t \in T$; therefore, the function X is a stochastic process.

(7) Finally, it can be concluded that the function X defined by formula (59) and considered in point (6) constitutes the desired almost certain F-solution of the random differential inclusion (5)-(6) with initial condition (7), generated by control (25), which ends the proof of Theorem III.1.

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隨機系統之最佳時間定位回饋控制器

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摘　要

本論文討[illegible]論的是針對隨機非連續多值之運動阻力模式進行最佳時間之定位控制，並以統策理論來深討利用確定回饋閉迴[illegible]路控制系統之控制架構。實驗之驗證，證實此控制系統之正確工作及一些優點特別是在強健性領域上。爲了證明[illegible]在控制設計上之正確性，假設描述系統動態之隨機非連續微分關係之解爲已知且具有已知的一些特徵。本論文亦[illegible]包含許多實際應用的建議。