Random time-optimal control for mechanical systems

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ABSTRACT. *Mechanical systems with complex and uncertain models of the resistance to motion are described using random discontinuous differential inclusions. Several practical concepts for sub-time-optimal feedback controllers that provide many applicational advantages have been investigated in this paper by generalizing the concept of the classic switching curve to the switching region.*

RÉSUMÉ. *Systèmes mécaniques avec les modèles complexes et incertains des mouvements de résistance décrits par inclusions différentielles discontinues aléatoires. Quelques conceptions pratiques pour régulateurs sous-optimales en temps qui permettent plusieurs avantages pratiques élaborés dans cet article en généralisant une idée de courbe de commutation classique pour une région de commutation.*

MOTS-CLÉS : *systèmes mécaniques, contrôle optimal en temps, inclusion différentielle discontinue aléatoire, régulateur sous-optimal.*

KEY WORDS : *mechanical systems, time-optimal control, random discontinuous differential inclusion, suboptimal feedback controller.*

1. Introduction

The task of minimum-time control has been the subject of a vast amount of scientific research since the very beginning of the application of optimization theory to the practice of control. The basic solutions of time-optimal control for mechanical systems with a single degree of freedom have been thoroughly developed and described in classic textbooks; see e.g. Section 7.2 of [ATH 66]. Even though the results obtained are relatively simple and supported by stimulating interpretations, the proposed structures have proven in practical applications to be very sensitive to model inaccuracies and uncertainties. This has provided the premise for a large amount of research, which has resulted in diverse concepts, such as sliding mode control [UTK 92], fuzzy control (e.g. [KUL 98b, 98c]), and other robust control techniques [FRI 96, ISI 95, KHA 96, SLO 91, WEI 91].

The main reasons for the inaccuracies and uncertainties occurring in the models of mechanical systems are associated with friction phenomena, especially dry friction and stiction effects. More and more precise friction models have become the subject of numerous interdisciplinary research projects (see [ARM 95], also for the extensive literature). Unfortunately, the more accurate among these models frequently contain so many parameters whose actual values cannot be specified experimentally, and increase in addition the dimensionality of the object to such an extent, that the construction of applicational control structures becomes essentially impossible.

As a result, a concept was proposed in paper [KUL 96a] based on the possibilities offered by the application of random discontinuous differential inclusions. (Differential inclusions [AUB 84, KIS 90] were introduced in the 1960s, as a consequence of the adoption of multivalued functions, increasingly useful in scientific experiments, for dynamical systems theory.) Thus the mathematical foundations for applications of random discontinuous differential inclusions to time-optimal control for mechanical systems with friction models, along with strict proofs of the theoretical aspects of this problem, were presented in a series of papers [KUL 96a, 96b, 96c, 96d]. This theoretical material will be used in the present article for a holistic presentation of the concept of applying this type of differential inclusions to the synthesis – convenient in practice – of suboptimal control structures in the presence of a complex and uncertain model of friction forces.

Namely, consider a single degree of freedom mechanical system, whose dynamics are described by the differential inclusion

$$\ddot{y}(t) \in H(\dot{y}(t), y(t), t) + u(t) \quad , \qquad [1]$$

where y denotes a position of the object, u is a bounded control variable (either a force or a torque depending on application), and the function H models the resistance to motion. If one omits this factor, i.e. when H is identically equal to zero, the above inclusion can be reduced to the classical differential equation that expresses the second law of Newtonian mechanics:

$$\ddot{y}(t) = u(t) \quad . \tag{2}$$

The essential element of model [1] is the multivalued (set-valued) function H describing resistance to motion. In the majority of applications these resistances are associated primarily with friction phenomena, which depend mainly on velocity; more generally they can depend on position and be non-autonomous as well, for example if there are spring forces and/or gravitational effects. This function can therefore be expressed in the form

$$H(\dot{y}(t), y(t), t) = v(\dot{y}(t), y(t), t)\, F(\dot{y}(t)) \quad , \tag{3}$$

where v is a continuous mapping, and F denotes a piecewise continuous function that may be additionally multivalued at the points of discontinuity. For the sake of illustration, a simple form of such a function could be

$$F(\dot{y}(t)) = \begin{cases} 1 & \text{if } \dot{y}(t) > 0 \\ [-s, s] & \text{if } \dot{y}(t) = 0 \\ -1 & \text{if } \dot{y}(t) < 0 \end{cases} \quad , \tag{4}$$

where the parameter $s > 1$ is related to static friction. Since the solution of this problem by deterministic methods, due to the very general conditions formulated above, has proven to be impossible without significant trivialization of the model, a probabilistic concept will be proposed here. The main assumption is a condition stating that the function v introduced in equation [3] represents the realization of a given stochastic process V with almost all the realizations being continuous and jointly bounded. Such a model regards as probabilistic uncertainty the dependence of resistance on a number of other factors, not only $\dot{y}(t)$, $y(t)$ and t, but also those which are usually omitted to simplify the model. The probabilistic concept naturally admits in addition the typical uncertainties, in the form of perturbations and noise occurring in real systems.

This paper is organized as follows. In the next Section 2, the mathematical foundations of random discontinuous differential inclusions are presented. The terms introduced here are used in the main theorem, presented in Section 3. The applicational conclusions – concepts for four suboptimal structures convenient in practice – are contained in the following Sections 4 and 5.

2. Foundations of differential inclusions

Let T be an interval with nonempty interior.
First, consider the deterministic differential inclusion

$$\dot{x}(t) \in G(x(t), t) \quad , \qquad [5]$$

where $G: \mathbb{R}^n \times T \rightarrow \mathscr{P}(\mathbb{R}^n)$, $x: T \rightarrow \mathbb{R}^n$, and $\mathscr{P}(A)$ denotes the set of subsets of A. In the case of the discontinuity of the mapping G, the so-called C-, F- and K-solutions, i.e. in Caratheodory, Filippov, and Krasovski senses [HAJ 79, KUL 96b], are the most frequently used. The first of these is applied when that mapping is discontinuous only with respect to the variable t. The derivative of the C-solution is dependent (besides the variable t) only on the present value of the solution. By contrast, in accordance with the necessity in contemporary engineering to take into account the influence of inaccuracies and approximations, the derivative of the K-solution depends on all the points in the neighborhood of the present value of this solution. Continuing the concept of taking errors into account, the F-solution also omits zero-measure sets, which are unimportant from a practical point of view.

Definition 1

The function x, absolutely continuous on every compact subinterval of the set T, is a solution of differential inclusion [5]:

– in the Caratheodory sense (C-solution), if it satisfies inclusion [5] almost everywhere in T,
– in the Filippov sense (F-solution), if

$$\dot{x}(t) \in \mathscr{F}[G](x(t), t) \quad \text{almost everywhere in } T \ , \qquad [6]$$

– in the Krasovski sense (K-solution), if

$$\dot{x}(t) \in \mathscr{K}[G](x(t), t) \quad \text{almost everywhere in } T \ , \qquad [7]$$

where the operators $\mathscr{F}$ and $\mathscr{K}$ are defined by

$$\mathscr{F}[G](x(t), t) = \bigcap_{e>0} \ \bigcap_{Z \subset \mathbb{R}^n : m(Z)=0} conv[G((x(t) + eB) \setminus Z, t)] \qquad [8]$$

$$\mathscr{K}[G](x(t), t) = \bigcap_{e>0} conv[G(x(t) + eB, t)] \quad , \qquad [9]$$

B denotes the open unit ball in the space $\mathbb{R}^n$, m is the Lebesgue measure, and $conv[C]$ means the convex closed hull of the set C. ■

Suppose also that $t_0 \in T$ and $x_0 \in \mathbb{R}^n$.

Definition 2

The C-, F- or K-solution of the deterministic differential inclusion [5] with the initial condition

$$x(t_0) = x_0 \qquad [10]$$

is unique, if all C-, F- or K-solutions, respectively, are identically equal functions.■

Proposition 3

(A) The C-solution is a K-solution (since $G(x(t),t) \in \mathscr{K}[G](x(t),t)$).
(B) The F-solution is a K-solution (because $\mathscr{F}[G](x(t),t) \subset \mathscr{K}[G](x(t),t)$). ■

Example 4

Suppose the differential equation with an initial condition

$$\dot{x}_1(t) = \begin{cases} 1 & \text{for } x_2(t) > 0 \\ -1 & \text{for } x_2(t) = 0 \\ 1 & \text{for } x_2(t) < 0 \end{cases}, \qquad x_1(0) = x_{01} \qquad [11]$$

$$\dot{x}_2(t) = \begin{cases} -1 & \text{for } x_2(t) > 0 \\ 0 & \text{for } x_2(t) = 0 \\ 1 & \text{for } x_2(t) < 0 \end{cases}, \qquad x_2(0) = x_{02}, \qquad [12]$$

where $x_{01}, x_{02} \in \mathbb{R}$ (Fig. 1). (Note that if the right-hand side of inclusion [5] is univalued, it can be reduced to a differential equation, which has been done here to make the example clearer.)

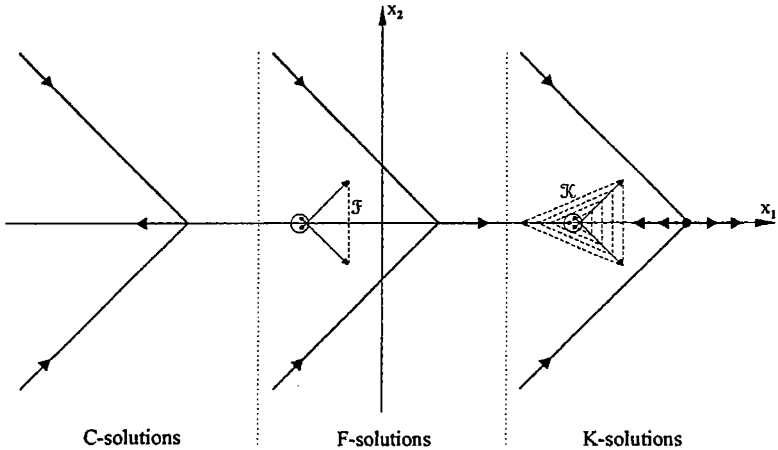

Figure 1. *Solutions of differential equation [11]–[12]*

First, the case $x_{02} = 0$ will be considered. The function $[x_{01} - t, 0]^T$ is then a C-solution, while the mapping $[x_{01} + t, 0]^T$ constitutes an F-solution (for the values of the operator $\mathscr{F}$, see Fig. 1). In turn, every absolutely continuous function of the form $[x_1(t), 0]^T$, such that $x_1(0) = x_{01}$ and the condition $|\dot{x}_1(t)| \leq 1$ is fulfilled at the points of existence of the derivative, represents a K-solution (see also Fig. 1).

In the case $x_{02} \neq 0$, however, the C-, F- and K-solutions are given, until the x_1 axis is crossed, as $[x_{01} + t, x_{02} - t \cdot \mathrm{sgn}(x_{02})]^T$. After the axis is reached, these solutions can be extended, in accordance with the rules presented in the preceding paragraph. ■

The above example indicates that there is a lack of relation between C- and F-solutions, whereas K-solutions often constitute an excessively broad class. Thus, in the consideration of differential inclusions (and equations) with a discontinuous right-hand side, a considerable difficulty is presented by the lack of any universal concept of a solution [KUL 96b]. Of course, the existence of C-, F- and K-solutions that are unique and equal to each other considerably simplifies further analysis.

The foregoing concepts of solutions will be generalized below to a random case. Such a generalization, however, is not unique. In this paper, the concept of almost certain solutions (with probability 1, first type) will be applied, because of its natural interpretation, which refers at a fixed random factor to the notions of the deterministic approach presented above.

Let (Ω, Σ, P) denote a probability space. (From a practical point of view, its completeness can be assumed without any loss in generality [RUD 74, Section 1.36].)

Consider the random differential inclusion

$$X(\omega,t) \in G(\omega, X(\omega,t),t) \quad , \qquad [13]$$

where $G:\Omega\times R^n \times T \to \mathscr{P}(\mathbb{R}^n)$ and X denotes an n-dimensional stochastic process (defined on the interval T).

Definition 5

A stochastic process X is an almost certain C-, F- or K-solution of the random differential inclusion [13] if almost all its realizations are C-, F- or K-solutions, respectively, of the corresponding deterministic differential inclusions obtained by fixing $\omega \in \Omega$. ■

Assume also that X_0 is an n-dimensional random variable.

Definition 6

The almost certain C-, F- or K-solution of the random differential inclusion [13] with the initial condition

$$X(\omega,t_0) = X_0(\omega) \qquad \text{for almost all } \omega \in \Omega \qquad [14]$$

is unique, if all almost certain C-, F- or K-solutions, respectively, are equivalent stochastic processes (i.e. for the processes $X^{\sim}$ and $X^{\approx}$ this means that $P(\{\omega \in \Omega: X^{\sim}(\omega,t) = X^{\approx}(\omega,t)\}) = 1$ for every $t \in T$). ■

The generalization of the concept of time-optimal control to random systems is not unique, either. From a practical point of view, it would be most useful to define a control that is only a function of time and the state in closed-loop systems, realizing the minimum expected value of the time to reach the target set. Unfortunately, such a formulation of the problem does not offer hope for its solution. In what follows, a different definition of the time-optimal control for random systems is formulated. This control, by analogy to the almost certain solution, will be called an almost certain time-optimal control.

Definition 7

Let $G:\Omega\times\mathbb{R}^n\times\mathbb{R}^m\times T \to \mathscr{P}(\mathbb{R}^n)$, U denote an m-dimensional stochastic process (defined on the interval T), and the differential inclusion

$$X(\omega,t) \in G(\omega, X(\omega,t), U(\omega,t), t) \quad , \qquad [15]$$

with the initial condition

$$X(\omega,t_0) = X_0(\omega) \qquad \text{for almost all } \omega \in \Omega \qquad [16]$$

describe the dynamics of a random system submitted to the control U. Then, the m-dimensional stochastic process U_o will be called an almost certain time-optimal control, if almost all its realizations are time-optimal controls (that is, they bring the state to the target set in a minimal and finite time) for proper deterministic systems obtained by fixing $\omega \in \Omega$. ■

Because a control thus defined is time-optimal for factors ω with probability 1, it ensures the realization of the minimum expected value of the time to reach the target set; however, it depends additionally on the random factor, which is obviously unavailable for the control algorithm. The result of this dependence stating the above control is difficult to apply directly, but may be a useful basis for the creation of technical constructions of suboptimal structures in which the direct dependency of the control function on the random factor has been eliminated. In the next section a theorem is presented regarding the almost certain time-optimal control for the mechanical systems given by equations [1] and [3]. Its characterization will make it possible in Section 4 to formulate conclusions of an applicational nature.

3. Almost certain time-optimal control

Below will be presented the theorem whose thesis constitutes a solution of the random time-optimal control problem considered in this paper.

Theorem 8

Assume:

(A) $t_0 \in \mathbb{R}$, $T = [t_0, \infty)$, $x_0 \in \mathbb{R}^2$, $v_-, v_+ \in \mathbb{R}$ such that $[v_-, v_+] \subset (-1,1)$;

(B) the origin of coordinates constitutes a target set;

(C) $U_a = \{u : T \to [-1,1]$, measurable $\}$ represents a set of admissible controls;

(D) $f : \mathbb{R} \to [-1,1]$ denotes a piecewise continuous function fulfilling locally a Lipschitz condition except at points of discontinuity, and $z \cdot f(z) \geq 0$ for every $z \in \mathbb{R}$; also, $F : \mathbb{R} \to \mathscr{P}([-1,1])$ is such that

$$F(z) = \begin{cases} f(z) & \text{if} \quad z \neq z_i \\ F_i & \text{if} \quad z = z_i \end{cases} \quad , \qquad [17]$$

where $z_i \in \mathbb{R}$, $F_i \subset [-1,1]$, and $i = 1, 2, \ldots, k$;

(E) (Ω, Σ, P) denotes a complete probability space;

(F) V is a real stochastic process with almost all realizations being continuous, and satisfying the boundary condition $V(\omega, t) \in [v_-, v_+]$ for $t \in T$;

(G) the random differential inclusion

$$X_1(\omega, t) = X_2(\omega, t) \quad [18]$$

$$X_2(\omega, t) \in U(\omega, t) - V(\omega, t)\, F(X_2(\omega, t)) \quad , \quad [19]$$

with the initial condition

$$\begin{bmatrix} X_1(\omega, t_0) \\ X_2(\omega, t_0) \end{bmatrix} = x_0 \quad \text{for almost all } \omega \in \Omega \quad [20]$$

describes the dynamics of the system submitted to the control U.

Then, there exists an almost certain time-optimal control U_o, whose realizations take on the values $+1$, -1, and have at most one point of discontinuity. This control generates a unique almost certain C-solution, which is also a unique almost certain F-solution and a unique almost certain K-solution. ■

The proof of this theorem can be found in paper [KUL 96a] together with article [KUL 96d]. The state space has been subdivided here into the disjoint sets R_+, R_-, Q_+, Q_-, and $\{(0,0)\}$ – see Fig. 2. Specifically, let K_{+-}, K_{++} denote sets of all states which can be brought to the origin by the control $U \equiv +1$, if $V \equiv v_-$ or $V \equiv v_+$, respectively; analogously K_{--} and K_{-+} for $U \equiv -1$, if $V \equiv v_-$ or $V \equiv v_+$, respectively. Moreover, let:

$$Q_+ = \{ [x_1, x_2]^T \in \mathbb{R}^2 \text{ such that there exist } [x_1', x_2]^T \in K_{++} \text{ and } [x_1'', x_2]^T \in K_{+-} \text{ with } x_1' \le x_1 \le x_1'' \} \quad [21]$$

$$Q_- = \{ [x_1, x_2]^T \in \mathbb{R}^2 \text{ such that there exist } [x_1', x_2]^T \in K_{--} \text{ and } [x_1'', x_2]^T \in K_{-+} \text{ with } x_1' \le x_1 \le x_1'' \} \quad [22]$$

$$R_+ = \{ [x_1, x_2]^T \in \mathbb{R}^2 \setminus Q \text{ such that there exist } [x_1', x_2]^T \in Q \text{ with } x_1 < x_1' \} \quad [23]$$

$$R_- = \{ [x_1, x_2]^T \in \mathbb{R}^2 \setminus Q \text{ such that there exist } [x_1', x_2]^T \in Q \text{ with } x_1' < x_1 \} \quad , \quad [24]$$

where $Q = Q_+ \cup \{[0,0]^T\} \cup Q_-$. Therefore, the sets K_{+-}, K_{++} represent all those states which can be brought to the origin of coordinates by the control $+1$, at the minimum and maximum values of resistance to motion. The set Q_+ contains intermediate points. The sets K_{--}, K_{-+} and Q_- for the control -1 may be interpreted analogously (Fig. 2).

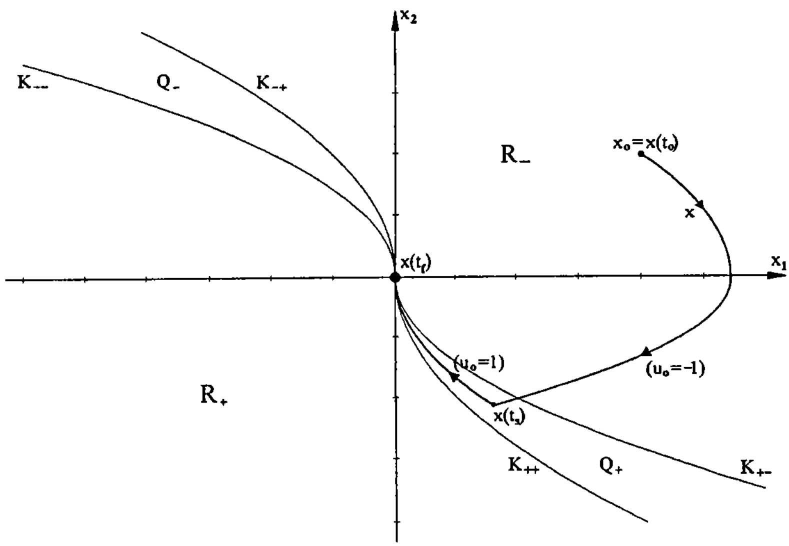

Figure 2. *Illustration of proof of Theorem 8*

Let the random factor ω first be fixed, and therefore the realization $V(\omega,\cdot)$ as well. If $x_0 \in R_-$, there exists t_s such that the unique and equal to each other C-, F- and K-solutions x generated by the control

$$u_o(t) = \begin{cases} -1 & \text{for } t \in [t_0, t_s) \\ +1 & \text{for } t \in [t_s, \infty) \end{cases} \tag{25}$$

reach the origin in the finite time t_f, with $t_0 < t_s < t_f$ and $x(t_s) \in Q_+$ (Fig. 2). Analogously, if $x_0 \in R_+$, there exists t_s such that the solutions generated by

$$u_o(t) = \begin{cases} +1 & \text{for } t \in [t_0, t_s) \\ -1 & \text{for } t \in [t_s, \infty) \end{cases} \tag{26}$$

reach the origin in the finite time t_f, with $t_0 < t_s < t_f$ and $x(t_s) \in Q_-$.

In the case that $x_0 \in Q_+$, the time-optimal control for the particular elements of conditions given above, or

$$u_o(t) = +1 \quad \text{for} \quad t \in [t_0, \infty) \quad . \qquad [27]$$

The case $x_0 \in Q_-$ is analogous. The counterpart of control [27] is then

$$u_o(t) = -1 \quad \text{for} \quad t \in [t_0, \infty) \quad . \qquad [28]$$

The proof of the time-optimality of the above controls has been based on the theory of differential inequalities. Finally, by a superposition of the corresponding mappings, it has been shown that, in the case of the control of the form [25] and [26], the time t_s is a random variable with bounded values, and also that the function defined as

$$U_o(\omega, \cdot) \equiv u_o \quad , \qquad [29]$$

where u_o has previously been given with the random factor $\omega \in \Omega$ fixed, constitutes a stochastic process, and therefore the almost certain time-optimal control. It can be proved in a similar manner that the families of unique and equal C-, F- and K-solutions generated by this control are stochastic processes [KUL 96a, KUL 96d].

The change of sign in the particular realizations of the control U_o (so-called switching of the control) can occur only when the system state belongs to the set (closed region) Q. For this reason it will be called a switching region.

Finally: the switching curve γ familiar from the classic case of minimum-time point-to-point transfer of a pure mass, i.e. double integrator system [ATH 66, Section 7.2], has been generalized by Theorem 8 to the switching region Q ($\gamma = Q$ when $v_- = v_+ = 0$). The function H introduced in formula [1], which represents the model of resistance to motion, has been decomposed into two factors: $F(\dot{y}(t))$ and $V(\omega, t)$. The former, a deterministic one, makes it possible to incorporate the properties of discontinuity and multivalency of friction phenomena. The latter one, thanks to its probabilistic nature, includes among other things approximations and identification errors (of the first factor also), the dependence of resistance on position, time and temperature, as well as perturbations and noise naturally occurring in real systems. The switching curve (even more general than γ, i.e. obtained with the condition $v_- = v_+$) implied by the first – deterministic – factor has been "blurred" by the second – random – to the switching region.

A detailed treatment, and in particular commentaries on the various assumptions, may be found in papers [KUL 96a, 96d].

4. Applicational conclusions: suboptimal control structures

As mentioned earlier, besides specific cases, the direct implementation of a system generating the almost certain time-optimal control encounters difficulties because of its dependence on the random factor, in fact unknown *a priori*. However, thanks to the results of Theorem 8, the presented material constitutes a useful basis for the creation of suboptimal control laws, in which such a dependence is removed.

Below four structures will be given, exhaustive combinations of the open and closed configuration, and cases where the properties of an actuator accept a full range of control values from the interval $[-1,1]$ or only its extreme values ± 1. In general outline, they constitute an extension of the deterministic "bang-bang" control, used in robotics for planning point-to-point displacement of robot manipulators [SCI 96].

Example 9

From Theorem 8 it results that U_o is a stochastic process with bounded values. There exists, then, its expected value, and, as a deterministic mapping with values in the interval $[-1,1]$, it can be used in the construction of a suboptimal control. ■

Example 10

For the fixed $x_0 \in R_- \cup R_+$ it has been shown [KUL 96a] that the change of sign in particular realizations of the process U_o is a random variable with bounded values. Its expected value therefore exists, and may be treated as the switching time of the suboptimal control of the form $(-1,+1)$ or $(+1,-1)$ respectively, which is then a deterministic function. The case $x_0 \in Q_- \cup Q_+$ can be formulated similarly. Note that, in contrast to Example 9, the actuator that generates only the extreme values of the set of admissible controls, i.e. $+1$ and -1, is sufficient for the practical realization of such a structure. ■

The controls presented above were defined in an open configuration. Closed-loop systems are preferable, however, given the requirements of engineering practice.

Example 11

The following concept will be based on the physical properties of friction phenomena, and in particular on the fact that the impact of resistance to motion on dynamical processes can be subject in certain conditions to averaging. Thus, after the sensitivity of the control system to the motion resistance value is analyzed in detail, the elements of statistical decision theory [BER 80] will be used for the synthesis of the feedback controlling structure.

The basic task of decision theory is the optimal selection of one element from among all possible decisions on the sole basis of probabilistic information about the state of nature, especially when its real state is unknown. Let the following be given:

(A) $\mathcal{N} \subset \mathbb{R}$ – a non-empty set of possible states of nature,

(B) $\mathcal{D} \subset \mathbb{R}$ – a non-empty set of possible decisions,

(C) the function

$$\ell : \mathcal{N} \times \mathcal{D} \to \mathbb{R} \cup \{\pm\infty\} \ , \qquad [30]$$

in which its value $\ell(n, d)$ is interpreted as losses resulting from making the decision d while in reality the state n is occurring.

The mapping $\ell_m : \mathcal{D} \to \mathbb{R} \cup \{\pm\infty\}$ given by the formula

$$\ell_m(d) = \sup_{n \in \mathcal{N}} \ell(n, d) \qquad [31]$$

is known as the minimax loss function. If additionally the probability space $(\mathcal{N}, \mathcal{S}, \mathcal{P})$ is defined on the set $\mathcal{N}$, and for every $d \in \mathcal{D}$ the integral $\int_{\mathcal{N}} \ell(n, d)\, d\mathcal{P}(n)$ exists, then the mapping $\ell_b : \mathcal{D} \to \mathbb{R} \cup \{\pm\infty\}$ given as

$$\ell_b(d) = \int_{\mathcal{N}} \ell(n, d)\, d\mathcal{P}(n) \qquad [32]$$

is called the Bayes loss function. Then, every element $d_m \in \mathcal{D}$ such that

$$\ell_m(d_m) = \inf_{d \in \mathcal{D}} \ell_m(d) \qquad [33]$$

is said to be a minimax decision, and analogously, every element $d_b \in \mathcal{D}$ fulfilling the condition

$$\ell_b(d_b) = \inf_{d \in \mathcal{D}} \ell_b(d) \qquad [34]$$

is called a Bayes decision, while the above procedures are known as the minimax and Bayes rules, respectively. Therefore, the Bayes rule minimizes the expected value of losses, whereas the minimax rule minimizes the greatest loss that may occur after a given decision is made.

In accordance with the concept of the suboptimal structure designed in this example, a particular case will now be considered of the probabilistic measure P connected with the stochastic process V, when it is concentrated on constant realizations (interpreted as average values of resistance to motion).

If the value of this constant realization is known and equals v, then according to the notations of Theorem 8

$$\nu_- = \nu_+ = \vartheta \quad , \tag{35}$$

therefore $K_{+-} = K_{++}$ and $K_{--} = K_{-+}$, that is, the switching region Q is confined to the curve whose shape is dependent on the parameter ϑ. If the value of its estimator $\hat{\vartheta}$ assumed in the feedback controller equations is equal to the value ϑ occurring in the system, i.e. in the case $\hat{\vartheta} = \vartheta$, the obtained control is time-optimal (Fig. 3). The state of the system is brought to the switching curve, and being permanently included there hereafter, it reaches the target in a minimal and finite time. The trajectory representative for the case $\hat{\vartheta} > \vartheta$ is shown in Fig. 4. As a result of its having oscillations around the target, over-regulations occur in the system. The state reaches the target in a finite time. Finally, Fig. 5 presents trajectories representative for the case $\hat{\vartheta} < \vartheta$. The target is reached in a finite time. Sliding trajectories appear in the system. In each case, the time to reach the target set increases from the optimal more or less proportionally to the difference between the values $\hat{\vartheta}$ and ϑ.

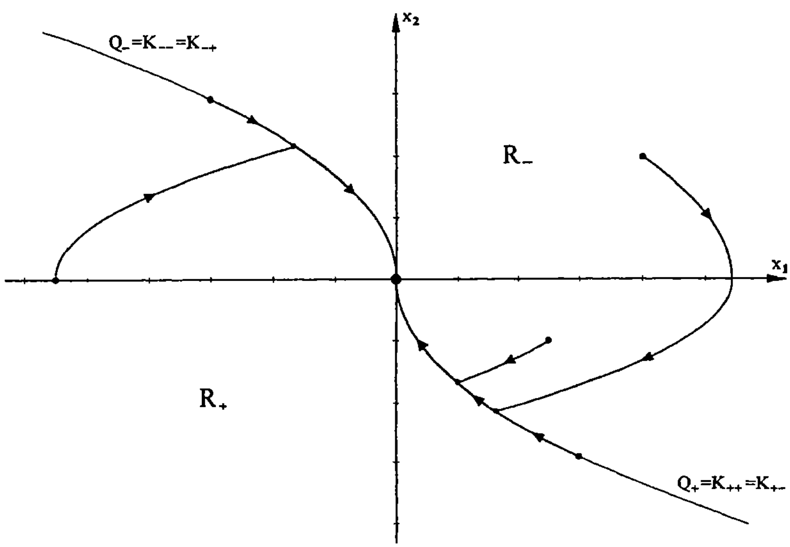

Figure 3. *Case* $\hat{\vartheta} = \vartheta$

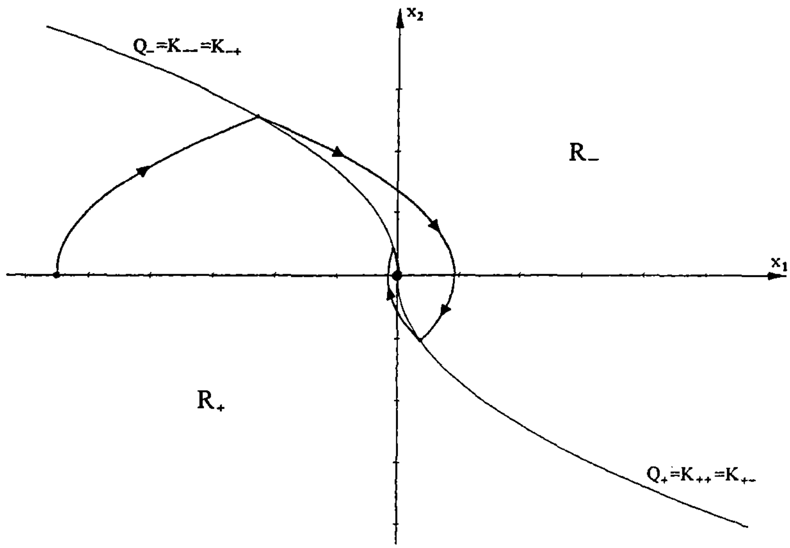

Figure 4. *Case* $\hat{\vartheta} > \vartheta$

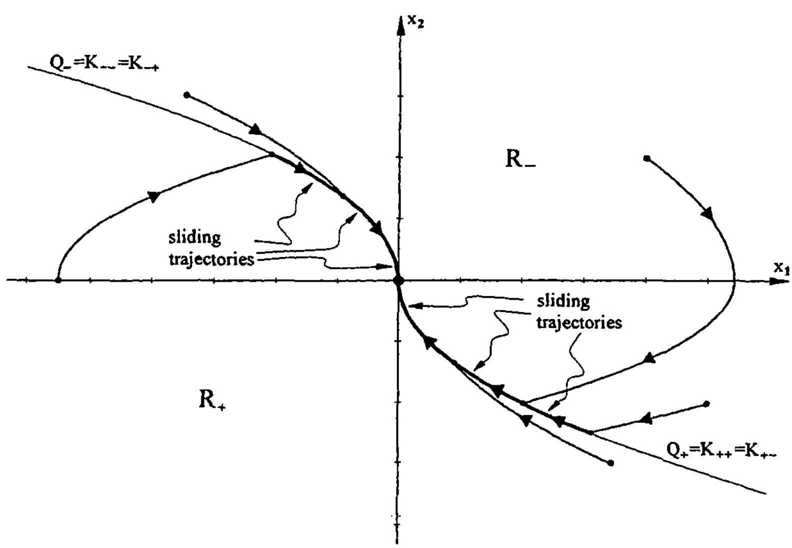

Figure 5. *Case* $\hat{\vartheta} < \vartheta$

In practice, the value of the parameter v is not known *a priori*; in the following considerations, it will be treated as a random variable. The elements of decision theory presented earlier, in conjunction with the results of the preceding sensitivity analysis, imply the following suggestions regarding the rules for determining the value of the estimator $\hat{v}$. This value will be treated as a decision, whereas the parameter v will be considered the state of nature. The loss function is associated with the increase in the time to reach the target when the control has been switched too soon or too late.

If over-regulations can be allowed in the controlled system, then the estimator $\hat{v}$ should be determined on the basis of the Bayes rule with real values for the loss function. Such a choice is possible because the calculation of the parameter $\hat{v}$ value that is either smaller than, equal to, or greater than v allows the system state to be brought to the target in a finite time (although this time increases along with the difference between the values $\hat{v}$ and v).

If, however, over-regulations are not allowed, this determination should be carried out on the basis of the minimax rule assuming infinite values of the loss function for $\hat{v} > v$. This enables the over-regulations to be avoided, because they occur only if $\hat{v} > v$.

In order to illustrate the above algorithm, let v be a random variable whose distribution has the density function h, with the support of the form $[v_-, v_+]$, while $[v_-, v_+] \subset (-1,1)$, and moreover, assume that it is continuous and positive in the interval (v_-, v_+). The loss function [30] will be described by the formula

$$l(\hat{v}, v) = \begin{cases} -p(\hat{v} - v) & \text{for } \hat{v} - v \le 0 \\ q(\hat{v} - v) & \text{for } \hat{v} - v \ge 0 \end{cases}, \qquad [36]$$

where $p, q \in \mathbb{R}^+ \cup \{\infty\}$; however, only one of them can be infinite. In the case of infinite value, let $\infty \cdot 0 = 0$.

According to the above assumptions, it is accepted that $\mathcal{N} = \mathcal{D} = [v_-, v_+]$.

With fixed $\hat{v}$, it results from dependencies [31] and [32] that

$$l_m(\hat{v}) = \max(\{-p(\hat{v} - v_+), q(\hat{v} - v_-)\}) \qquad [37]$$

$$l_b(\hat{v}) = \int_{v_-}^{\hat{v}} q(\hat{v} - v)\, h(v)\, dv - \int_{\hat{v}}^{v_+} p(\hat{v} - v)\, h(v)\, dv \quad . \qquad [38]$$

If $q = \infty$, then thanks to equation [37] it can be obtained that the infimum of the function l_m on the set $\mathcal{D}$ is assumed by

$$\hat{v} = v_- \quad . \qquad [39]$$

(For the purposes of the next section, it will still be noted that, in the case when $p = \infty$, the above infimum is realized by

$$\hat{v} = v_+ \quad .) \tag{40}$$

Now, thanks to the assumption of the continuity of the function h, the mapping l_b is differentiable in the interval (v_-, v_+); therefore, from formula [38] one obtains

$$l_b'(\hat{v}) = p \int_{v_+}^{\hat{v}} h(v)\, dv + q \int_{v_-}^{\hat{v}} h(v)\, dv \tag{41}$$

and analogously

$$l_b''(\hat{v}) = (p + q)\, h(\hat{v}) \quad . \tag{42}$$

By using equality [41] the equivalence of the following conditions can be proved by means of elementary transformations:

$$l_b'(\hat{v}) = 0 \tag{43}$$

$$\int_{v_-}^{\hat{v}} h(v)\, dv = \frac{p}{p + q} \quad . \tag{44}$$

Dependence [42] implies that the function l_b'' is positive in the interval (v_-, v_+); therefore, the function l_b is here strictly convex. Because $0 < p/(p+q) < 1$, equation [44], equivalent to condition [43], is fulfilled only in one point; here, the function l_b assumes its minimum, global in the set $\mathcal{D} = [v_-, v_+]$, thanks to the continuity of this function in the points v_- and v_+. A more general proof of the foregoing, omitting the condition of the continuity of the function h, is found in article [KUL 99].

The value $\hat{v}$ satisfying condition [44] constitutes the quantile of order $p/(p+q)$. A usable algorithm for calculating it, based on neural networks technique, has been presented in papers [KUL 98d, SCH 97].

To summarize, in accordance with the considerations stated before, if over-regulations cannot be accepted in the controlling system, and the value of the estimator $\hat{v}$ should be determined due to the minimax rule with infinite values of the loss function for $q = \infty$, then it is obtained from formula [39]. If, however, over-regulations are permissible, and thus the value $\hat{v}$ may be determined by using the Bayes rule for the real parameters p and q, then equation [44] is applied.

If one possesses the value $\hat{v}$ thus obtained, the feedback controller equations can be calculated. Accordingly, for the example function F described by formula

[4], the set $Q_{+} = K_{++} = K_{+-}$ is defined on the plane $x_1 - x_2$ by the dependence

$$x_1 = \frac{x_2^2}{2(1+\hat{\sigma})} \quad \text{for} \quad x_2 \in (-\infty, 0) \ , \qquad [45]$$

while $Q_{-} = K_{--} = K_{-+}$ is given by

$$x_1 = -\frac{x_2^2}{2(1+\hat{\sigma})} \quad \text{for} \quad x_2 \in (0, \infty) \ . \qquad [46]$$

The sets R_{-} and R_{+} result from formulas [23] and [24]. Finally, the suboptimal control can be defined in a closed-loop manner by the following equality:

$$U_s(\omega,t) = \begin{cases} -1 & \text{if } [X_1(\omega,t), X_2(\omega,t)]^T \in (R_{-} \cup Q_{-}) \\ 0 & \text{if } [X_1(\omega,t), X_2(\omega,t)]^T \in \{[0,0]^T\} \\ +1 & \text{if } [X_1(\omega,t), X_2(\omega,t)]^T \in (R_{+} \cup Q_{+}) \end{cases} \ . \qquad [47]$$

Figure 6 provides an illustration of the results obtained in the above example. A detailed description of the foregoing structure, along with the requisite mathematical proofs, can be found in article [KUL 96c]. ■

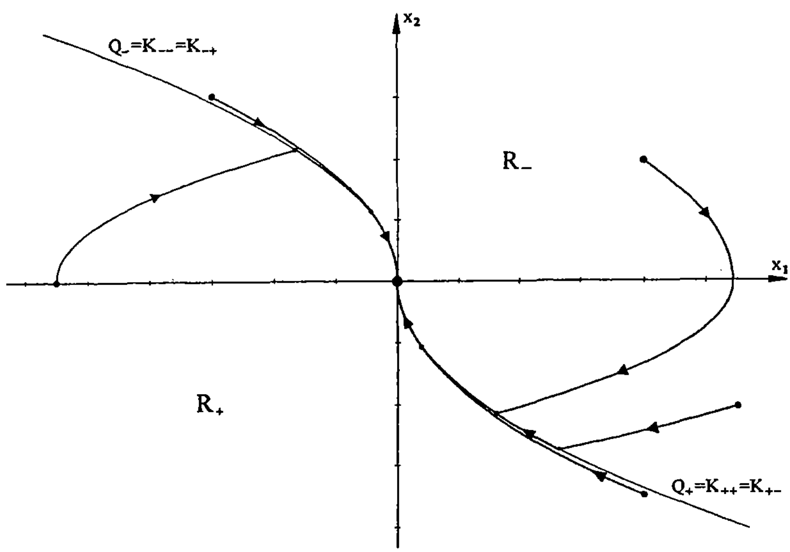

Figure 6. *Trajectories generated by feedback controller designed in Example 11*

Example 12

In the case when the used actuator allows all control values from the interval $[-1,1]$, the structure worked out in Example 11 can be modified to eliminate frequent switchings between the values $+1$ and -1 occurring on the sliding trajectories. In mechanical systems such switchings often have a negative impact on the actuator life and may excite vibrations in elastic transmissions; hence they should be avoided. This goal will be achieved by making the control designed in the previous Example continuous.

Thus, in addition to the constant $\hat{\upsilon}$ introduced in Example 11, let the parameter $\hat{\upsilon}$ also be given, with $-1<\tilde{\upsilon}<\hat{\upsilon}$. Just as in Theorem 8, let K_{+-}, K_{++} denote sets of all states which can be brought to the origin by the control $U\equiv+1$, if $V\equiv\tilde{\upsilon}$ or $V\equiv\hat{\upsilon}$, respectively; analogously K_{--} and K_{-+} for $U\equiv-1$, if $V\equiv\tilde{\upsilon}$ or $V\equiv\hat{\upsilon}$, respectively (Fig. 7). The sets Q_+, Q_- and R_-, R_+ remain unchanged, and are accordingly given by dependencies [21]-[24]. The suboptimal control is now defined by the formula

$$U_s(\omega,t)=\begin{cases}-1 & \text{if } [X_1(\omega,t),X_2(\omega,t)]^T\in R_- \\ -z(-x_1,-x_2) & \text{if } [X_1(\omega,t),X_2(\omega,t)]^T\in Q_- \\ 0 & \text{if } [X_1(\omega,t),X_2(\omega,t)]^T\in\{[0,0]^T\} \\ z(x_1,x_2) & \text{if } [X_1(\omega,t),X_2(\omega,t)]^T\in Q_+ \\ +1 & \text{if } [X_1(\omega,t),X_2(\omega,t)]^T\in R_+\end{cases}, \qquad [48]$$

where the function $z:\mathbb{R}^2\to\mathbb{R}$ takes on the value $1-\upsilon_+ +\upsilon_-$ on the sets K_{+-} and K_{--}, after which it increases linearly to the value 1 on the sets K_{++} and K_{-+}, i.e.

$$z(x_1,x_2)=\frac{\upsilon_+-\upsilon_-}{k_{++}(x_2)-k_{+-}(x_2)}(x_1-k_{++}(x_2))+1 , \qquad [49]$$

while the mappings k_{++} and $k_{+-}:(0,\infty)\to\mathbb{R}$ are defined naturally as

$$x_1=k_{++}(x_2)\Leftrightarrow[x_1,x_2]^T\in K_{++} \qquad [50]$$

$$x_1=k_{+-}(x_2)\Leftrightarrow[x_1,x_2]^T\in K_{+-} . \qquad [51]$$

For the example function F given by formula [4], the sets K_{-+} and K_{--} are defined on the plane x_1-x_2 by the equations

$$x_1=\frac{x_2^2}{2(1+\hat{\upsilon})} \quad \text{for } x_2\in(-\infty,0) \qquad [52]$$

$$x_1 = \frac{x_2^2}{2(1+\tilde{\sigma})} \quad \text{for} \quad x_2 \in (-\infty, 0) \ , \qquad [53]$$

whereas K_{++} and K_{+-} by the dependencies

$$x_1 = -\frac{x_2^2}{2(1+\hat{\sigma})} \quad \text{for} \quad x_2 \in (0, \infty) \qquad [54]$$

$$x_1 = -\frac{x_2^2}{2(1+\tilde{\sigma})} \quad \text{for} \quad x_2 \in (0, \infty) \ , \qquad [55]$$

respectively.

The trajectories generated by the controlling structure worked out above are shown in Fig. 7. They resemble the results achieved on a bob-sled track thanks to the appropriate modeling of its shape.

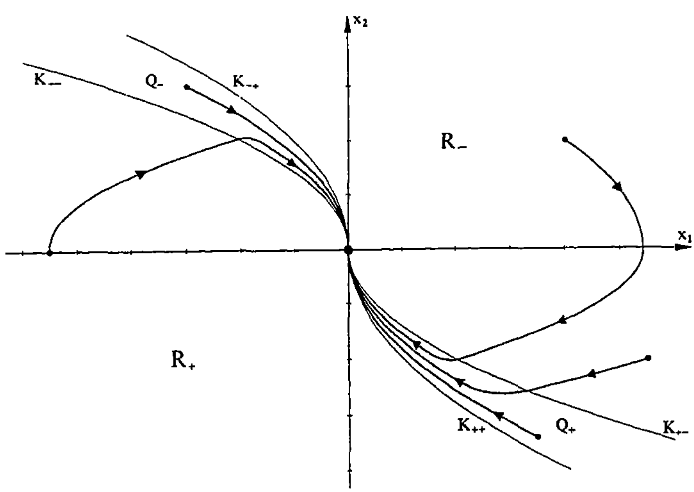

Figure 7. *Trajectories generated by feedback controller designed in Example 12*

The value of the parameter $\tilde{\sigma}$ can be determined heuristically. In general, the difference $\hat{\sigma} - \tilde{\sigma}$ should be proportional to the delay occurring in the object. ■

5. Generalizations

Theorem 8 was formulated in its basic version. The resulting suboptimal control structures can easily be supplemented, however, with a number of new aspects that recur in engineering practice. The first to be considered will be the issue of velocity limitation, often essential in many applications. Without this aspect, the basic time-optimal control law for a mechanical system, particularly a robot manipulator, may lead to unacceptably high velocities, if the distance between the initial and target points is too long [SCI 96]. The second problem to be taken up here is a generalization of the target set to any point in the state space, especially including cases where the target position must be reached with a given velocity.

Namely, taking into account the condition of limiting the velocity to the value b, where $b > 0$, therefore introducing the assumption

$$|X_2(\omega,t)| \leq b \quad \text{for all} \quad t \in T \quad \text{and almost all} \quad \omega \in \Omega \ , \qquad [56]$$

causes the realizations of the almost certain time-optimal control of the form [25] to be modified into

$$u_o(t) = \begin{cases} -1 & \text{for} \ \ t \in [t_0, t_1) \\ v(t) \cdot F(-b) & \text{for} \ \ t \in [t_1, t_2) \\ +1 & \text{for} \ \ t \in [t_2, \infty) \end{cases} \ , \qquad [57]$$

while $t_0 \leq t_1 \leq t_2 \leq t_f < \infty$, $x(t_2) \in Q_+$ and if $t_1 \neq t_2$ then likewise $x(t) \in \mathbb{R} \times \{-b\}$ for $t \in [t_1, t_2]$. (The symbol v denotes the realization of the stochastic process V corresponding to the fixed random factor $\omega \in \Omega$, i.e. $v \equiv V(\omega, \cdot)$.) Analogously, the realizations described by formula [26] are changed to

$$u_o(t) = \begin{cases} +1 & \text{for} \ \ t \in [t_0, t_1) \\ v(t) \cdot F(b) & \text{for} \ \ t \in [t_1, t_2) \\ -1 & \text{for} \ \ t \in [t_2, \infty) \end{cases} \ , \qquad [58]$$

while $t_0 \leq t_1 \leq t_2 \leq t_f < \infty$, $x(t_2) \in Q_-$ and if $t_1 \neq t_2$ then likewise $x(t) \in \mathbb{R} \times \{b\}$ for $t \in [t_1, t_2]$. In the case of the closed-loop suboptimal structure from Example 11, the control can be defined by the dependence

$$U_s^{\sim}(\omega,t) = \begin{cases} -1 & \text{for} \ \ \mathrm{X}_2(\omega,t) \geq b \\ U_s(\omega,t) & \text{for} \ \ \mathrm{X}_2(\omega,t) \in (-b,b) \\ +1 & \text{for} \ \ \mathrm{X}_2(\omega,t) \leq -b \end{cases} \ , \qquad [59]$$

where U_s is given by formula [47]. The trajectories generated by control [] are shown in Fig. 8. The next one, Fig. 9, illustrates the concept of taking into ount the condition of limiting the velocity in the case of the structure from Exam 12. Like the constant $\tilde{\sigma}$, the parameter $\tilde{b}$ is dependent on the delay occurring the object.

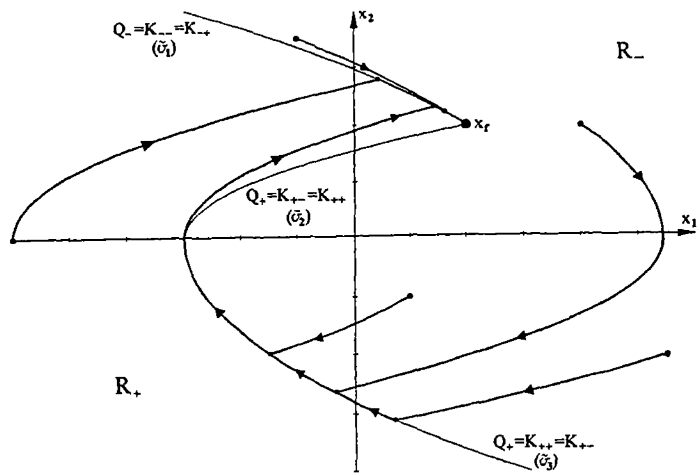

Figure 8. *Trajectories generated by feedback controller designed in Exampl1 for case of limited velocity*

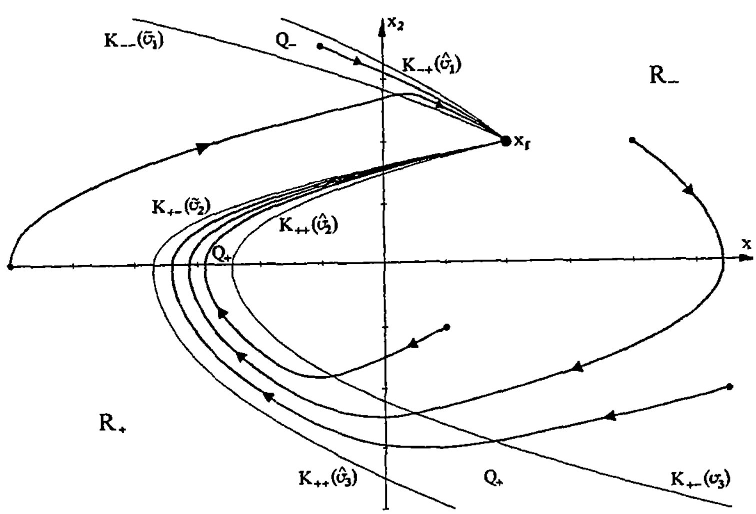

Figure 9. *Trajectories generated by feedback controller designed in Exampl2 for case of limited velocity*

The generalization of the target set to any point $x_f = [x_{f1}, x_{f2}]^T \in \mathbb{R}^2$ causes no change in the thesis of Theorem 8. If $x_{f2} = 0$, then the closed-loop structures presented in Examples 11 and 12 remain the same, due to the possibility of performing the simple transformation

$$X_1^* \equiv X_1 - x_{f1} \ . \qquad [60]$$

Only in the case when $x_{f2} \neq 0$ does it become necessary to introduce a modification, as a consequence of the differing results of the sensitivity analysis [KUL 96c, 98a]. The switching region (curve) Q obtained in Example 11 should now be divided into three parts, so that the target set and the intersection with the axis x_1 are the dividing points. For each of these parts, the values of the parameter $\hat{\upsilon}$, denoted here as $\hat{\upsilon}_1$, $\hat{\upsilon}_2$, $\hat{\upsilon}_3$ (Fig. 10), are determined in a different manner: $\hat{\upsilon}_1$, according to the minimax rule with infinite values of the loss function for $\hat{\upsilon}_1 > \upsilon_1$; the parameter $\hat{\upsilon}_2$, using the minimax rule but with infinite values for this function when $\hat{\upsilon}_2 < \upsilon_2$; the parameter $\hat{\upsilon}_3$ by the Bayes rule with real values of the loss function. (Thus, in the case of the loss function given by formula [36], the estimators $\hat{\upsilon}_1$, $\hat{\upsilon}_2$, $\hat{\upsilon}_3$ should be calculated from formulas [39], [40], and [44].) The trajectories generated by the control thus defined are shown in Fig 10, while a detailed description of the above concept is found in works [KUL 96c, 98a]. From the analysis presented there, it can also be inferred that the generalization of the target set in the case of the structure of Example 12 requires similar changes. The parameters $\hat{\upsilon}$ and $\tilde{\upsilon}$ are represented in the corresponding parts by $\hat{\upsilon}_1$, $\hat{\upsilon}_2$, $\hat{\upsilon}_3$ and $\tilde{\upsilon}_1$, $\tilde{\upsilon}_2$, $\tilde{\upsilon}_3$ while their values should be determined on the basis of the minimax rule with infinite values of the loss function for $\hat{\upsilon}_1 > \upsilon_1$, $\hat{\upsilon}_2 < \upsilon_2$ and by the Bayes rule with real values of this function (therefore, in the case of the loss function [36], using formulas [39], [40], and [44] consecutively). The parameters $\tilde{\upsilon}_1$, $\tilde{\upsilon}_2$, $\tilde{\upsilon}_3$ fulfilling the inequalities $-1 < \tilde{\upsilon}_1 < \hat{\upsilon}_1$, $\hat{\upsilon}_2 < \tilde{\upsilon}_2 < 1$ and $-1 < \tilde{\upsilon}_3 < \hat{\upsilon}_3$ are determined heuristically. An illustration of the suboptimal structure thus obtained is provided by Fig. 11 (note a small modification in the definitions of the sets K_{+-} and K_{++}, made here for consistency with formula [49]).

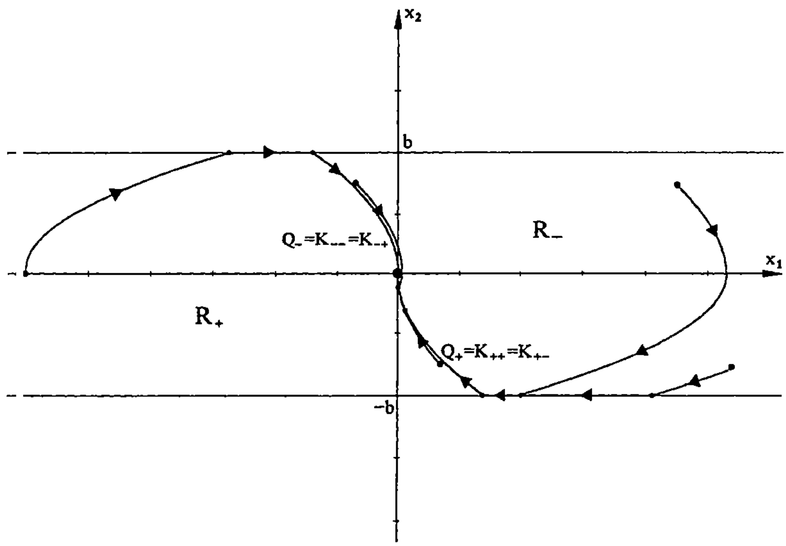

Figure 10. *Trajectories generated by feedback controller designed in Example 11 after generalization of target set*

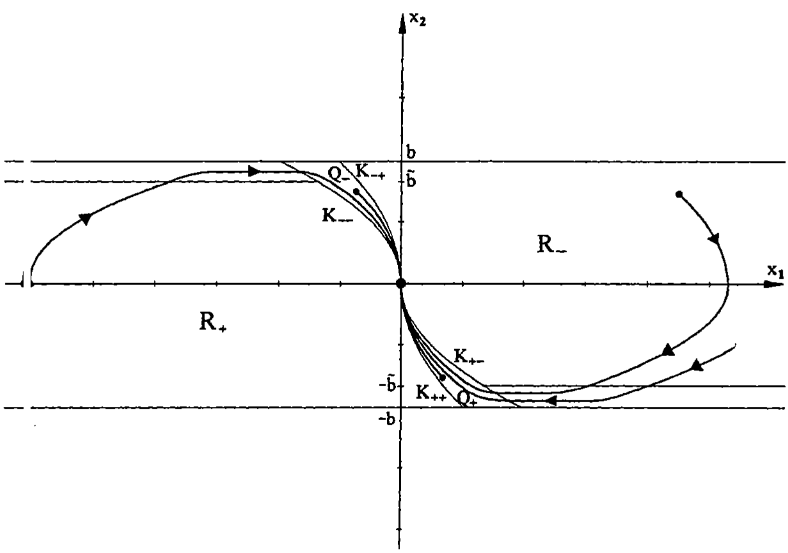

Figure 11. *Trajectories generated by feedback controller designed in Example 12 after generalization of target set*

In the case of the majority of the above suboptimal structures no almost certain C-solution exists, whereas the almost certain K-solution, as well as the even less general almost certain F-solution, is nonunique. The proof of their existence therefore requires the use of advanced mathematical theory, e.g. measurable selectors [KUL 96c].

6. Final comments

Mechanical systems with complex and uncertain models of the resistance to motion are described in this paper using random discontinuous differential inclusions. Several practical concepts for sub-time-optimal feedback controllers have been investigated here by generalizing the concept of the classic switching curve to the switching region.

The correct operation of the above controllers has been checked empirically. The system was observed to operate properly, provided only that the values of the resistance to motion not exceed the assumed support of the density function (see also Fig. 6-11, which was obtained with very diversified data). However, the time to reach the target set was the shorter, the more precise was the identification of the probabilistic data on the object. In a case where the probabilistic characteristics are non-autonomous, there appeared the suggestion of a corresponding adaptational structure. A broader discussion of the results of the empirical verification is contained in book [KUL 98a].

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