Challenges in restoring images corrupted by Poisson-Gaussian noise

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OBSERVATIONS

1. Noise in bright area
2. BUT ALSO in dark area
3. Light intensity decreases along time

MACROconfocal laser scanning microscope (Leica TCS-LSI) - cross-section rhizome of Convallaria majalis (Lily of the Valley), by Gilbert Engler INRA Sophia Antipolis, France.

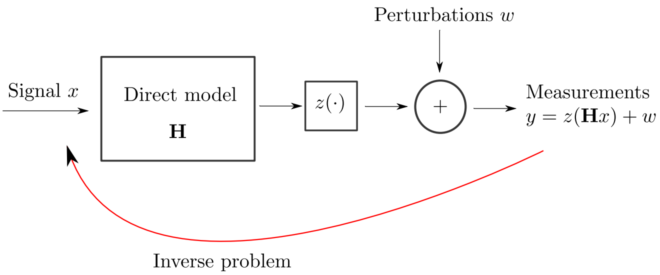
OBSERVATIONS

1. Noise in bright area
2. BUT ALSO in dark area
3. Light intensity decreases along time

MACROconfocal laser scanning microscope (Leica TCS-LSI) - image of Arabidopsis Thaliana seedlings,

by Gilbert Engler INRA Sophia Antipolis, France.

Objective



Inverse problem

Problem oriented towards reversal to cause-effect sequence i.e. Find $\boldsymbol{x}$ based on observations $\boldsymbol{y}$

MAXIMUM A POSTERIORI PROBABILITY ESTIMATOR

Maximum likelihood estimate

$$\boldsymbol{x} = \underset{\boldsymbol{x}}{\operatorname{argmax}}\, p(\boldsymbol{x} \mid \boldsymbol{y})$$

MAP

$$\underset{\boldsymbol{x}}{\operatorname{argmax}} \frac{p_{\boldsymbol{Y}}(\boldsymbol{y}|\boldsymbol{x})p_{\boldsymbol{X}}(\boldsymbol{x})}{p_{\boldsymbol{Y}}(\boldsymbol{y})} \Leftrightarrow$$
$$\underset{\boldsymbol{x}}{\operatorname{argmax}}\, p_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{x})p_{\boldsymbol{X}}(\boldsymbol{x}) \Leftrightarrow$$
$$\underset{\boldsymbol{x}}{\operatorname{argmax}}\, \log(p_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{x})) + \log\left(p_{\boldsymbol{X}}(\boldsymbol{x})\right)$$

- $-\log(p_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{x}))$ - an appropriate measure of dissimilarity between $\boldsymbol{x}$ and $\boldsymbol{y}$ (so-called data fidelity function)
- $-\log\left(p_{\boldsymbol{X}}(\boldsymbol{x})\right)$ - regularization function incorporating a priori information and guaranteeing the stability of the solution.

DEGRADATION MODEL

$$\boldsymbol{y} = \alpha \boldsymbol{z}(\boldsymbol{x}) + \boldsymbol{w}$$

Observations

$\boldsymbol{y} \in \mathbb{R}^S$

Degradation model

Poisson noise

$Z_s(\boldsymbol{x}) \sim \mathcal{P}([\boldsymbol{H}\boldsymbol{x}]_s))$

$\boldsymbol{z}(\boldsymbol{x}) = \big(z_s(\boldsymbol{x})\big)_{1\leq s\leq S}$ - realization of $Z_s(\boldsymbol{x})$

$\boldsymbol{H} \in [0,+\infty)^{S\times N}$ - point spread function

$\boldsymbol{x} \in [0,+\infty)^{N}$ - original signal

$\alpha \in \mathbb{R}$ - scaling parameter

$$\boldsymbol{y} = \alpha\, \boldsymbol{z}(\boldsymbol{x}) + \boldsymbol{w}$$

Observations

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DEGRADATION MODEL

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Gaussian noise

$W_s \sim \mathcal{N}(c, \sigma^2)$

$\boldsymbol{w} = (w_s)_{1\leq s\leq S}$ - realization of W_s

$c \in \mathbb{R}$ - mean $\quad \sigma^2$ - variance

Observations

$\boldsymbol{y} \in \mathbb{R}^S$

POISSON-GAUSSIAN DISTRIBUTION

Distribution through convolution product

Random variable: $\boldsymbol{Y} = \boldsymbol{Z} + \boldsymbol{W}$

Distribution: $p_{\boldsymbol{Y}}(\cdot \mid \boldsymbol{x}) = p_{\boldsymbol{Z}}(\cdot \mid \boldsymbol{x}) * p_{\boldsymbol{W}}$

♣

Thus

$$p_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{x}) = \prod_{s=1}^{S} \left(\sum_{n=0}^{+\infty} \frac{e^{-[\boldsymbol{H}\boldsymbol{x}]_s}([\boldsymbol{H}\boldsymbol{x}]_s)^n}{n!} \frac{e^{-\frac{1}{2\sigma^2}(y_s - c - \alpha n)^2}}{\sqrt{2\pi\sigma^2}} \right)$$

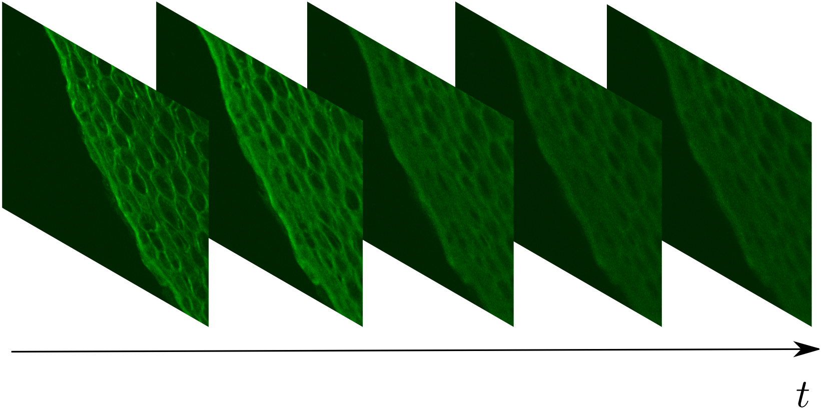
Challenge:

- To identify the noise parameters α, σ^2, c
- To restore the original signal $\boldsymbol{x}$ using MAP

Assumption: Linear operator $\boldsymbol{H}$ is assumed to be known

Poisson-Gaussian noise parameter estimation

TIME SAMPLES - BLEACHING EFFECT



Bleaching effect - process of intensity time decay usually modeled with an exponentially decreasing function

Dependence between noise model and photobleaching

We adopted the same model as the one presented in [Rodrigues *et al.* 2009]

- Signal independent noise sources remain the same
- Intrinsic luminous intensity of fluorophore decays exponentially with time
- Fluorophore behaviour is not uniform across a sample.

As a result:

- The Gaussian part of the model remains time-independent
- The exponential decay is put inside the Poisson model.
- We do not assume a unique decay rate for all pixels

MODEL

$\forall s \in \{1, \dots, S\}$
$\forall t \in \{1, \dots, T\}$

$$R_{s,t} = \alpha Q_{s,t} + W_{s,t}$$

Observations

$r = (r_{s,t})_{1 \le s \le S, 1 \le t \le T}$
$R = (R_{s,t})_{1 \le s \le S, 1 \le t \le T}$

MODEL

$\forall s \in \{1, \ldots, S\}$
$\forall t \in \{1, \ldots, T\}$

Poisson noise -

$Q_{s,t} \sim \mathcal{P}(u_s e^{-k_s t})$
$\alpha \in \mathbb{R}$ - scaling par.
$(u_s)_{1 \leq s \leq S} \geq 0$ - "clean" image
$(k_s)_{1 \leq s \leq S} \geq 0$ - bleaching decay

$$R_{s,t} = \alpha Q_{s,t} + W_{s,t}$$

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Gaussian noise

$W_{s,t} \sim \mathcal{N}(c, \sigma^2)$
$c \in \mathbb{R}$ - mean σ^2 - variance

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Gaussian noise

$W_{s,t} \sim \mathcal{N}(c, \sigma^2)$
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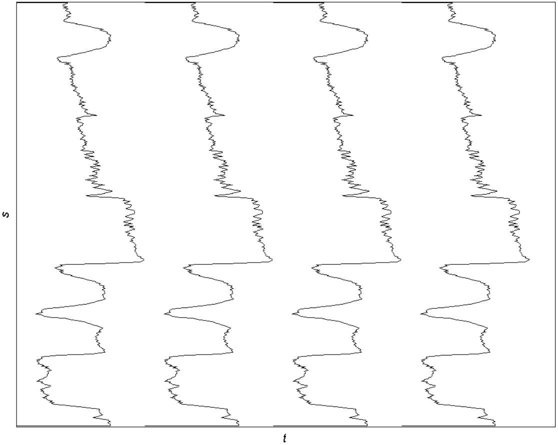
Problem

Find $\theta = (u, k, \alpha, c, \sigma^2)$

Example

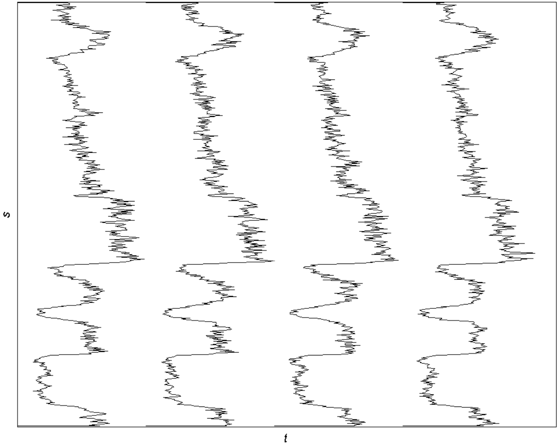


$U_{s,t}$

$T = 4$
$S = 512$

Original signal

Example

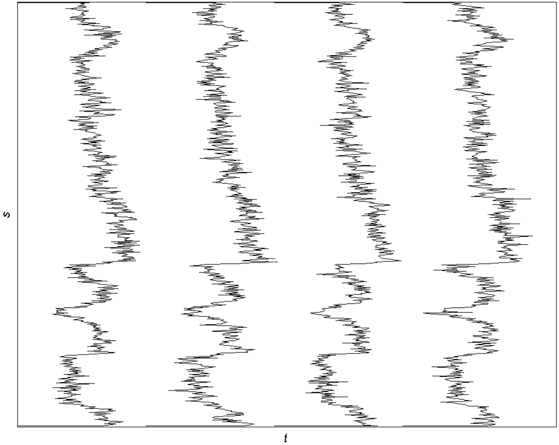


$Q_{s,t}$

$T = 4$
$S = 512$

Signal corrupted by Poisson noise

Example

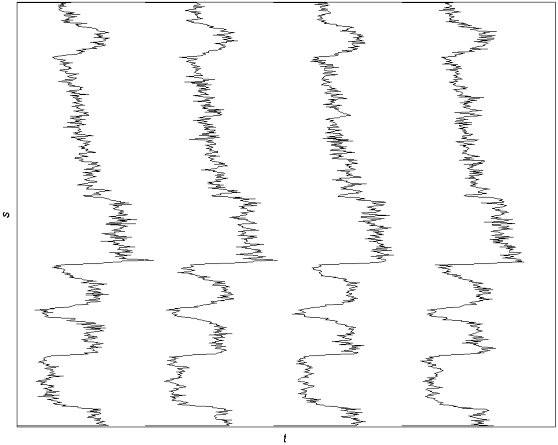


$U_{s,t} + W_{s,t}$

$T = 4$
$S = 512$

Signal corrupted by Gaussian noise

Example



$$\alpha Q_{s,t} + W_{s,t}$$

$$T = 4$$
$$S = 512$$

Signal corrupted by Poisson and Gaussian noises

Cumulants

$$\kappa_n[R_{s,t}] = \alpha^n \kappa_n[Q_{s,t}] + \kappa_n[W_{s,t}]$$

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Cumulant of order n

Cumulants

$$\kappa_n[R_{s,t}] = \alpha^n \kappa_n[Q_{s,t}] + \kappa_n[W_{s,t}]$$

Then:

- mean value

$$\kappa_1[R_{s,t}] = \mathsf{E}[R_{s,t}] = \alpha e^{-k_s t} u_s + c$$

- variance

$$\kappa_2[R_{s,t}] = \mathsf{Var}[R_{s,t}] = \alpha^2 e^{-k_s t} u_s + \sigma^2$$

- higher-order cumulants

$$\kappa_n[R_{s,t}] = \alpha^n e^{-k_s t} u_s, \qquad n \geq 3$$

Problem formulation

Using $\mathsf{E}[R_{s,t}] = \alpha e^{-k_s t} u_s + c$

we have: $R_{s,t} = a_s \, e^{-k_s t} + c + E_{s,t}$

$(a_s = \alpha u_s)_{1 \leq s \leq S}$

$(E_{s,t})_{1 \leq s \leq S, 1 \leq t \leq T}$ - independent zero-mean random variables.

Optimization criteria

$$(\widehat{a}, \widehat{k}, \widehat{c}) = \underset{a,k,c}{\operatorname{argmin}} \sum_{s=1}^{S} \sum_{t=1}^{T} \left(r_{s,t} - c - a_s e^{-k_s t} \right)^2$$

Traditional approach

Rewrite the problem as

$$\underset{k\in[0,+\infty[^S}{\text{minimize}} \ \psi(k)$$

where $(\forall k = (k_s)_{1\leq s\leq S} \in [0,+\infty[^S)$

$$\psi(k) = \min_{a\in\mathbb{R}^S, c\in\mathbb{R}} \sum_{s=1}^{S}\sum_{t=1}^{T}\left(r_{s,t} - c - a_s e^{-k_s t}\right)^2.$$

Problem

For large-size problems (S large), the minimization of ψ requires to solve a large dimensional non-convex minimization problem.

Reformulation in the product space $\mathbb{R}^S \times \mathbb{R}^S$

$$\underset{\substack{(c_1,\ldots,c_S)\in\mathbb{R}^S\\(x_1,\ldots,x_S)\in\mathbb{R}^S}}{\text{minimize}} \sum_{s=1}^{S} \varphi_s(c_s, x_s) + \iota_D(c_1, \ldots, c_S)$$

where $(\forall (c_s, x_s) \in \mathbb{R}^2)$

$$\varphi_s(c_s, x_s) = \begin{cases} \min\limits_{a_s\in\mathbb{R}} \sum_{t=1}^{T} \left(r_{s,t} - c_s - a_s x_s^t\right)^2 & \text{if } x_s \in [0,1] \\ +\infty & \text{otherwise,} \end{cases}$$

$$D = \{(c_1, \ldots, c_S) \in \mathbb{R}^S \mid c_1 = \cdots = c_S\}$$

and ι_D is the indicator function of D defined as

$$(\forall c = (c_1, \ldots, c_S) \in \mathbb{R}^S) \qquad \iota_D(c_1, \ldots, c_S) = \begin{cases} 0 & \text{if } c \in D \\ +\infty & \text{otherwise.} \end{cases}$$

Solution

Estimates $\widehat{c}$ and $(\widehat{x}_s)_{1\leq s\leq S}$ are obtained using the Douglas-Rachford algorithm [Borwein *et al.* 2011]. Require to compute proximity operators (prox_{ι_D} and $\mathrm{prox}_{\gamma\varphi_s}$).

♣

Where

$$\mathrm{prox}_f : \mathbb{R}^N \to \mathbb{R}^N : \boldsymbol{x} \to \mathrm{argmin}_{\boldsymbol{y}\in\mathbb{R}^N} f(\boldsymbol{y}) + \frac{1}{2}\|\boldsymbol{x}-\boldsymbol{y}\|^2$$

$$(\forall (c_s)_{1\leq s\leq S} \in \mathbb{R}^S) \qquad \mathrm{prox}_{\iota_S}(c_1,\ldots,c_S) = \frac{c_1+\cdots+c_S}{S}(1,\ldots,1).$$

Then, we deduce:

$$(\forall s \in \{1,\ldots,S\}) \qquad \widehat{a}_s = \widehat{x}_s^2 \frac{1-\widehat{x}_s^2}{1-\widehat{x}_s^{2T}} \sum_{t=1}^{T} (r_{s,t}-\widehat{c})\widehat{x}_s^t.$$

Remaining parameters

Closed-form expressions

$$\widehat{\alpha} = \frac{ST\sum_{s=1}^{S}\widehat{a}_s f_s - \sum_{s=1}^{S} e_s \sum_{s=1}^{S}\widehat{a}_s\overline{\omega_s}}{ST\sum_{s=1}^{S}\widehat{a}_s^2\overline{\omega_s^2} - (\sum_{s=1}^{S}\widehat{a}_s\overline{\omega_s})^2},$$

$$(\forall s \in \{1,\ldots,S\}) \qquad \widehat{u}_s = \frac{\widehat{a}_s}{\widehat{\alpha}}.$$

$$\widehat{\sigma}^2 = \sum_{(s,t)} \left(e_{s,t} - \widehat{\alpha}\widehat{a}_s\widehat{x}_s^t\right)/ST$$

Where

$$\overline{\omega_s} = \sum_{t=1}^{T} x_s^t,\ \overline{\omega_s^2} = \sum_{t=1}^{T} x_s^{2t}$$

$$e_s = \sum_{t=1}^{T} e_{s,t},\ f_s = \sum_{t=1}^{T} \widehat{x}_s^t e_{s,t}$$

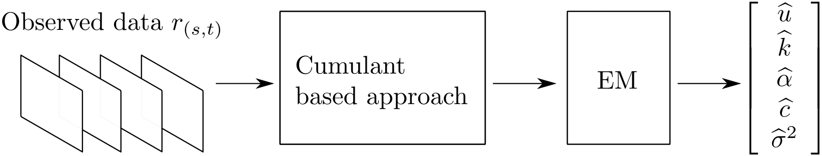
$$e_{s,t} = (r_{s,t} - \widehat{a}_s\widehat{x}_s^t - \widehat{c})^2$$

Refined estimation

Cumulant based approach -summary

- Cumulant approach estimates vector of unknown parameters θ
- Unknown parameters are not jointly estimated
- Possible large error propagation

Refined estimation



EM ALGORITHM [JEZIERSKA *et al.* 2011]

Joint density function $f_R(r \mid u, k, \alpha, c, \sigma)$ for Poisson + Gaussian:

$$\frac{1}{(\sqrt{2\pi}\sigma)^{ST}} \prod_{s=1}^{S} e^{-Tu_s} \prod_{t=1}^{T} \sum_{q_{s,t}=1}^{+\infty} e^{-\frac{(r_{s,t}-\alpha q_{s,t}-c)^2}{2\sigma^2}} \frac{(u_s e^{-k_s t})^{q_{s,t}}}{q_{s,t}!}$$

Expectation step

$$J(\theta \mid \theta^{(n)}) = \mathsf{E}_{Q|R=r,\theta^{(n)}}[\ln p_{R,Q}(R, Q \mid \theta)]$$

R - observations, Q - missing data ; θ - vector of parameters

Maximization step

$$(\forall n \in \mathbb{N}) \qquad \theta^{(n+1)} = \underset{\theta}{\operatorname{argmin}} - J(\theta \mid \theta^{(n)})$$

EM Maximization step

Update of $(\forall s \in \{1,\dots,S\}) \quad k_s^{(n+1)} = -\ln x^{(n+1)}$ **where** $x^{(n+1)}$ **is the solution in** $(0,1)$ **of the polynomial equation:**

$$
\begin{aligned}
&(1 + Tx^{T+1} - (T+1)x^T) \sum_{t=1}^{T} \mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] \\
&= (1 - x - x^T + x^{T+1}) \sum_{t=1}^{T} t\,\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}]
\end{aligned}
$$

solved with Halley's algorithm

Update in a closed form: $u_s^{(n+1)}$, $\alpha^{(n+1)}$, $c^{(n+1)}$, $\sigma^{(n+1)}$ assuming known conditional mean $\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}]$ and $\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}^2]$ computed in EM Expectation Step.

EM Expectation step

$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] = \sum_{q_{s,t}=1}^{+\infty} q_{s,t}\mathsf{P}(Q_{s,t} = q_{s,t} \mid R = r, \theta^{(n)})$$

EM EXPECTATION STEP

$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] = \sum_{q_{s,t}=1}^{+\infty} q_{s,t} \mathsf{P}(Q_{s,t} = q_{s,t} \mid R = r, \theta^{(n)})$$

After some calculations, we have:

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$$\zeta_{s,t}^{(n)} = \sum_{q_{s,t}=1}^{+\infty} e^{-\frac{(r_{s,t}-\alpha^{(n)}q_{s,t}-c^{(n)})^2}{2(\sigma^2)^{(n)}}} \frac{(u_s^{(n)}e^{-k_s^{(n)}t})^{q_{s,t}}}{(q_{s,t}-1)!}$$

EM Expectation step

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Problem: Infinite sums

Solution: Adaptive truncation technique

$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}^2]$ is computed similarly

TRICKS FOR THE TRUNCATION

$$\Pi_{s,t}(\theta, b, q_{s,t}) = \exp\left(-\frac{(r_{s,t} - \alpha(q_{s,t} + b) - c)^2}{2\sigma^2}\right) \frac{(u_s e^{-k_s t})^{q_{s,t}+b}}{q_{s,t}!}$$

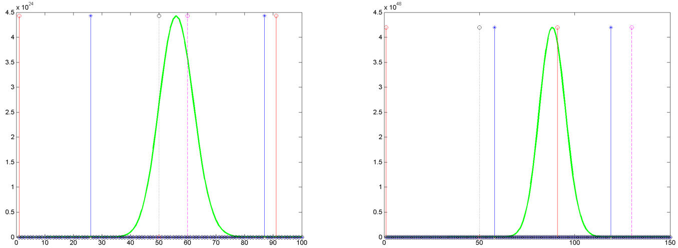

Figure: $\Pi_{s,t}(\theta, 0, 0, q_{s,t})$ as a function of $q_{s,t}$ for $r_{s,t} = 50$ (green) for θ: $\alpha = 1$, $c = 0$, $\sigma^2 = 100$. (left) $u_s = 60$ and (right) $u_s = 130$. Proposed bounds in blue. Bounds proposed in [Benvenuto *et al.* 2008] in red. Black dotted line: $r_{s,t}$; pink one: u_s.

Proposed Algorithm

Step 1: Cumulant Method

$\theta^{(1)} \leftarrow \left(u^{(1)}, k^{(1)}, \alpha^{(1)}, c^{(1)}, (\sigma^2)^{(1)}\right)$

Step 2: EM Algorithm

for $n \leftarrow 1$ to N **do**

Expectation step

$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] \leftarrow \frac{\zeta_{s,t}^{(n)}}{\eta_{s,t}^{(n)}}$$

$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}^2] \leftarrow \frac{\xi_{s,t}^{(n)}}{\eta_{s,t}^{(n)}}$$

Maximization step

$$\theta^{(n+1)} \leftarrow \left(u^{(n+1)}, k^{(n+1)}, \alpha^{(n+1)}, c^{(n+1)}, (\sigma^2)^{(n+1)}\right)$$

end for

PROPOSED ALGORITHM

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for $n \leftarrow 1$ to N **do**

Expectation step

$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] \leftarrow \frac{\zeta_{s,t}^{(n)}}{\eta_{s,t}^{(n)}}$$

$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}^2] \leftarrow \frac{\xi_{s,t}^{(n)}}{\eta_{s,t}^{(n)}}$$

Maximization step

$\theta^{(n+1)} \leftarrow \left(u^{(n+1)}, k^{(n+1)}, \alpha^{(n+1)}, c^{(n+1)}, (\sigma^2)^{(n+1)}\right)$

end for

Synthetic data results

Evaluation criteria

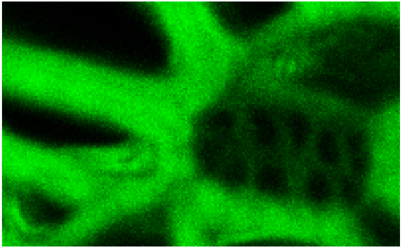
$$\mathrm{SNR} = 10\log_{10}\left(\frac{(ST)^{-1}\sum_{(t,s)}\left(a_s e^{-k_s t}\right)^2}{\sum_{(t,s)}\left(a_s e^{-k_s t} - \widehat{a}_s e^{-\widehat{k}_s t}\right)^2}\right)$$

Results

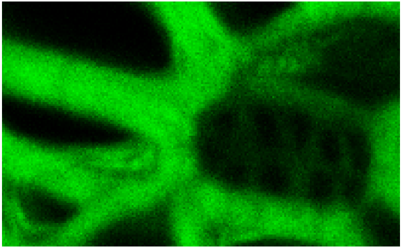
Statistics over 50 noise realizations.

Method.	$\hat{\sigma}$		$\hat{c}$		$\hat{\alpha}$		$\overline{\mathrm{SNR}}$
	bias	std	bias	std	bias	std	
Init.	357.5	3.1	1.9	1.0	−0.3	0.4	39.5
EM	2.9	0.9	1.4	0.8	−0.3	0.4	39.7

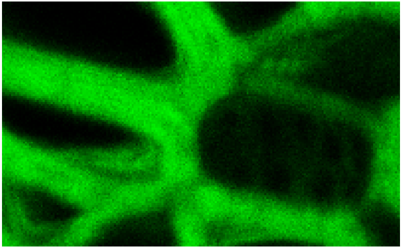
Table: Experiment conditions: $T = 180$, $S = 200$, $\alpha = 30$, $\sigma^2 = 100$, $c = 10$, $u_s \in [0, 150]$, $k_s \in [10^{-4}, 10^{-3}]$



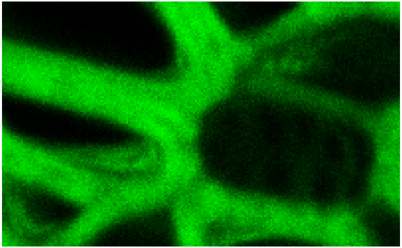
(a) $R_{s,1}$



(b) $R_{s,45}$

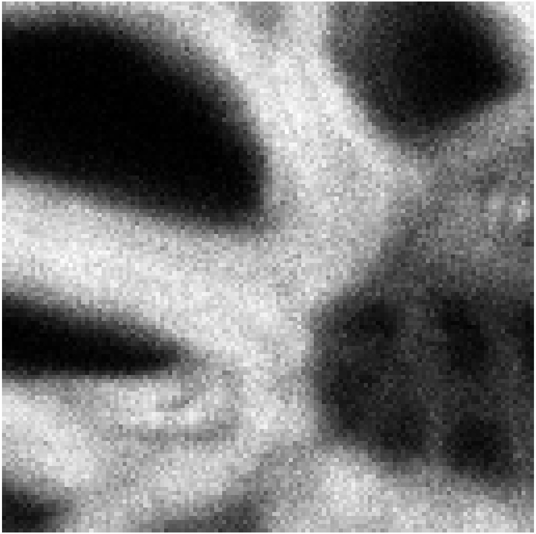


(c) $R_{s,90}$



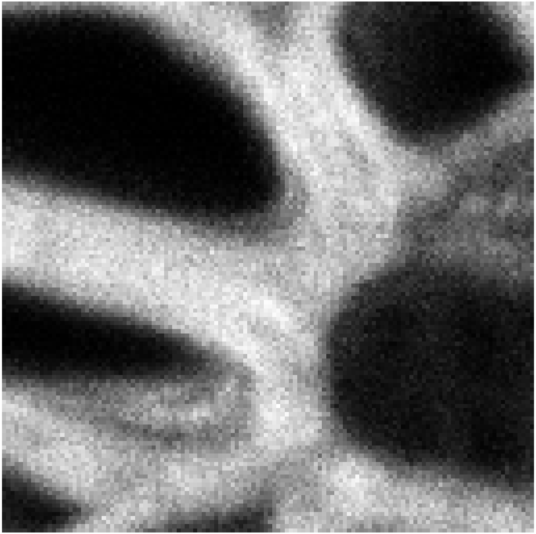
(d) $R_{s,180}$

VISUAL RESULTS



Original image $r_{s,1}$

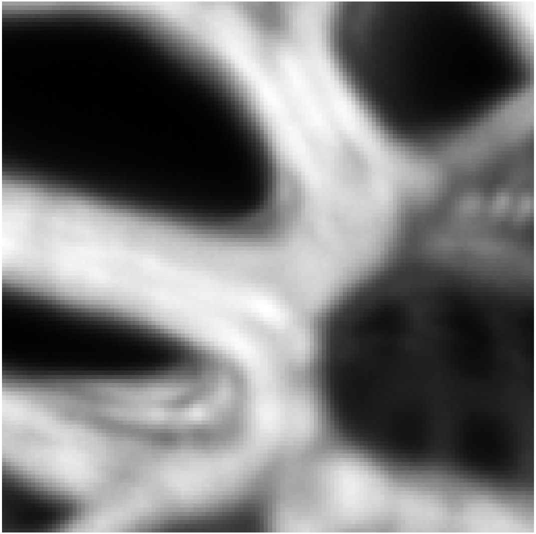
Visual results



Original image $r_{s,1}$

Original image $r_{s,180}$

VISUAL RESULTS

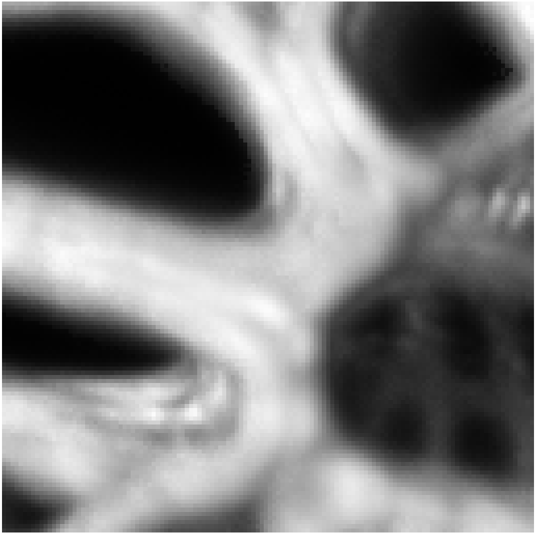


Original image $r_{s,1}$

Original image $r_{s,180}$

Mean over $T = 180$ realizations

Visual results



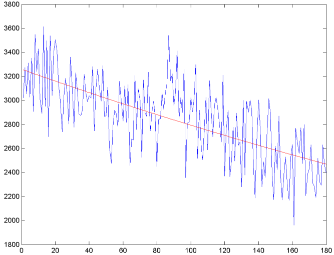
Original image $r_{s,1}$

Original image $r_{s,180}$

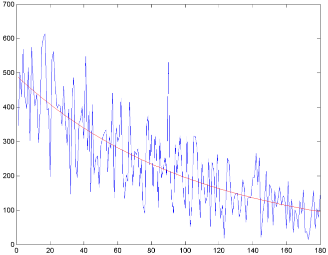
Mean over $T = 180$ realizations

Reconstructed image
Parameters:
$\widehat{\alpha} = 25.8$
$\widehat{c} = 8$
$\widehat{\sigma}^2 = 119$
$\widehat{u}_s \in [0, 147]$
$\widehat{k}_s \in [0, 3.9 \times 10^{-6}]$

Time variations



(a) $25.8 \times 126\, e^{-5.7\times 10^{-7} t} + 8$



(b) $25.8 \times 19\, e^{-3.6\times 10^{-6} t} + 8$

Figure: (a,b) illustrate time variations for fixed s. True data $R_{s,t}$ plotted in blue and estimated time curve (using formula $\widehat{\alpha}\widehat{u}_s e^{-\widehat{k}_s t} + \widehat{c}$) plotted in red.

Restoration of images corrupted by Poisson-Gaussian noise

Background

- Previously proposed strategies:
  grounded on any approximations of the noise statistics, not robust to numerical errors, cannot cover a wide range of priors
- Proximal methods:
  flexible, with guaranteed convergence, splitting methods, applicable to convex optimization problems

Proximity operator of function f

$$\mathrm{prox}_f : \mathbb{R}^N \to \mathbb{R}^N : \boldsymbol{x} \to \mathrm{argmin}_{\boldsymbol{y}\in\mathbb{R}^N} f(\boldsymbol{y}) + \frac{1}{2}\|\boldsymbol{x}-\boldsymbol{y}\|^2$$

- f - lower semi-continuous proper convex function

POISSON-GAUSSIAN DISTRIBUTION

Poison Gaussian neg log likelihood

$$-\sum_{s=1}^{S} \log \left(\sum_{n=0}^{+\infty} \frac{e^{-[\boldsymbol{H}\boldsymbol{x}]_s}([\boldsymbol{H}\boldsymbol{x}]_s)^n}{n!} \frac{e^{-\frac{1}{2\sigma^2}(y_s - c - \alpha n)^2}}{\sqrt{2\pi\sigma^2}} \right)$$

♣

Challenge:

To establish one of the following properties :

- explicit form of proximity operator of $-\log(p_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{x}))$
- μ-Lipschitz differentiability of $-\log(p_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{x}))$

Theorem

The function $g(\boldsymbol{x}) = -\log(p_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{x}))$ is convex and μ-Lipschitz differentiable on $[0, +\infty)^N$

$$\mu = \|\boldsymbol{H}\|^2 \left(1 - e^{-\frac{1}{\sigma^2}}\right) \max_{s\in\{1,\dots,S\}} e^{\frac{2(y_S - c) - 1}{\sigma^2}}$$

Gradient on the positive orthant

$$(\forall \boldsymbol{x} \in [0, +\infty)^N) \qquad \nabla g(\boldsymbol{x}) = \boldsymbol{H}^\top (\mathbf{1} - \boldsymbol{u}(\boldsymbol{H}\boldsymbol{x}))$$

$$\forall_{\boldsymbol{\xi}=(\xi_s)_{1\le s\le S}} \ \boldsymbol{u}(\boldsymbol{\xi}) = \left(\frac{\Xi(\xi_s, y_s - c - 1)}{\Xi(\xi_s, y_s - c)}\right)_{1\le s\le S}$$

$$\forall_{(\xi,\upsilon)\in\mathbb{R}^2} \ \Xi(\xi, \upsilon) = \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} e^{-\frac{1}{2\sigma^2}(\upsilon - \alpha n)^2}$$

Note: gradient of the Poisson- Gaussian negative log-likelihood involves infinite sums and cannot be computed exactly

Problem formulation

Find

$$\hat{\boldsymbol{x}} \in \operatorname{argmin} \quad f(\boldsymbol{x})$$

Where

$$f(\boldsymbol{x}) = h(\boldsymbol{x}) + r_0(\boldsymbol{x}) + \sum_{m=1}^{M} r_m(\boldsymbol{L}_m \boldsymbol{x})$$

- $r_m(\boldsymbol{L}_m \boldsymbol{x})$ - convex regularization term with linear operator $\boldsymbol{L}_m \in \mathbb{R}^{P_m \times N}$
- $r_0(\boldsymbol{x})$ - indicator function of a closed convex subset of $[0, +\infty)^N$
- $h(\boldsymbol{x})$ - for non-negative values defined as $-\log(p_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{x}))$ and which takes a quadratic form on $(-\infty, 0]^N$.

Primal-dual splitting approach [Combettes and Pesquet, 2011]

Initialization: $x_0 \in \mathbb{R}^N$, $(\forall m \in \{1,\dots,M\})$ $v_{m,0} \in \mathbb{R}^{P_m}$

for $k = 0, \dots$ **do**

$$y_{1,k} = x_k - \gamma\left(\nabla h(x_k) + \sum_{m=1}^M L_m^\top v_{m,k}\right) + a_k$$

$$p_{1,k} = \operatorname{prox}_{\gamma r_0}(y_{1,k})$$

for $m = 1, \dots, M$ **do**

$$y_{2,m,k} = v_{m,k} + \gamma L_m x_k$$

$$p_{2,m,k} = y_{2,m,k} - \gamma \operatorname{prox}_{\gamma^{-1} r_m}(\gamma^{-1} y_{2,m,k})$$

$$q_{2,m,k} = p_{2,m,k} + \gamma L_m p_{1,k}$$

$$v_{m,k+1} = v_{m,k} - y_{2,m,k} + q_{2,m,k}$$

end for

$$q_{1,k} = p_{1,k} - \gamma\left(\nabla h(p_{1,k}) + \sum_{m=1}^M L_m^\top p_{2,m,k}\right) + c_k$$

$$x_{k+1} = x_k - y_{1,k} + q_{1,k}$$

end for

Primal-dual splitting approach [Combettes and Pesquet, 2011]

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$$\boldsymbol{q}_{2,m,k} = \boldsymbol{p}_{2,m,k} + \gamma \boldsymbol{L}_m \boldsymbol{p}_{1,k}$$
$$\boldsymbol{v}_{m,k+1} = \boldsymbol{v}_{m,k} - \boldsymbol{y}_{2,m,k} + \boldsymbol{q}_{2,m,k}$$

end for

$$\boldsymbol{q}_{1,k} = \boldsymbol{p}_{1,k} - \gamma\left(\nabla h(\boldsymbol{p}_{1,k}) + \sum_{m=1}^M \boldsymbol{L}_m^\top \boldsymbol{p}_{2,m,k}\right) + \boldsymbol{c}_k$$
$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \boldsymbol{y}_{1,k} + \boldsymbol{q}_{1,k}$$

end for

Primal-dual splitting approach [Combettes and Pesquet, 2011]

Initialization: $\boldsymbol{x}_0 \in \mathbb{R}^N$, $(\forall m \in \{1,\ldots,M\})$ $\boldsymbol{v}_{m,0} \in \mathbb{R}^{P_m}$
for $k = 0,\ldots$ **do**

$\quad \boldsymbol{y}_{1,k} = \boldsymbol{x}_k - \gamma \left(\nabla h(\boldsymbol{x}_k) + \sum_{m=1}^{M} \boldsymbol{L}_m^\top \boldsymbol{v}_{m,k}\right) + \boldsymbol{a}_k$

$\quad \boldsymbol{p}_{1,k} = \mathrm{prox}_{\gamma r_0}(\boldsymbol{y}_{1,k})$

$\quad$ **for** $m = 1,\ldots,M$ **do**

$\quad\quad \boldsymbol{y}_{2,m,k} = \boldsymbol{v}_{m,k} + \gamma \boldsymbol{L}_m \boldsymbol{x}_k$

$\quad\quad \boldsymbol{p}_{2,m,k} = \boldsymbol{y}_{2,m,k} - \gamma\, \mathrm{prox}_{\gamma^{-1} r_m}(\gamma^{-1}\boldsymbol{y}_{2,m,k})$

$\quad\quad \boldsymbol{q}_{2,m,k} = \boldsymbol{p}_{2,m,k} + \gamma \boldsymbol{L}_m \boldsymbol{p}_{1,k}$

$\quad\quad \boldsymbol{v}_{m,k+1} = \boldsymbol{v}_{m,k} - \boldsymbol{y}_{2,m,k} + \boldsymbol{q}_{2,m,k}$

$\quad$ **end for**

$\quad \boldsymbol{q}_{1,k} = \boldsymbol{p}_{1,k} - \gamma \left(\nabla h(\boldsymbol{p}_{1,k}) + \sum_{m=1}^{M} \boldsymbol{L}_m^\top \boldsymbol{p}_{2,m,k}\right) + \boldsymbol{c}_k$

$\quad \boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \boldsymbol{y}_{1,k} + \boldsymbol{q}_{1,k}$

end for

CONVERGENCE

Assumptions:

❶ f: coercive, i.e. $\lim_{\|\boldsymbol{x}\|\to+\infty} f(\boldsymbol{x}) = +\infty$

❷ $\forall_{m\in\{1,\dots,M\}}\ r_m$: finite valued

❸ $\gamma \in [\epsilon, (1-\epsilon)/\beta]$, $\epsilon \in (0, 1/(\beta+1))$, $\beta = \mu + \sqrt{\sum_{m=1}^{M} \|\boldsymbol{L}_m\|^2}$

❹ $(\boldsymbol{a}_k)_{k\in\mathbb{N}}$ and $(\boldsymbol{c}_k)_{k\in\mathbb{N}}$: absolutely summable sequences

Result

There exists a minimizer $\overline{\boldsymbol{x}}$ of $f(\boldsymbol{x})$
s. t. $(\boldsymbol{x}_k)_{k\in\mathbb{N}}$ and $(\boldsymbol{p}_{1,k})_{k\in\mathbb{N}}$ converge to $\overline{\boldsymbol{x}}$

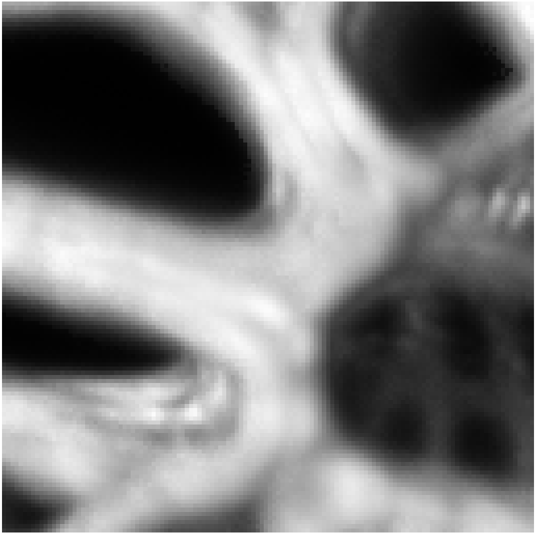
♣

RESULTS

Total variation penalization:

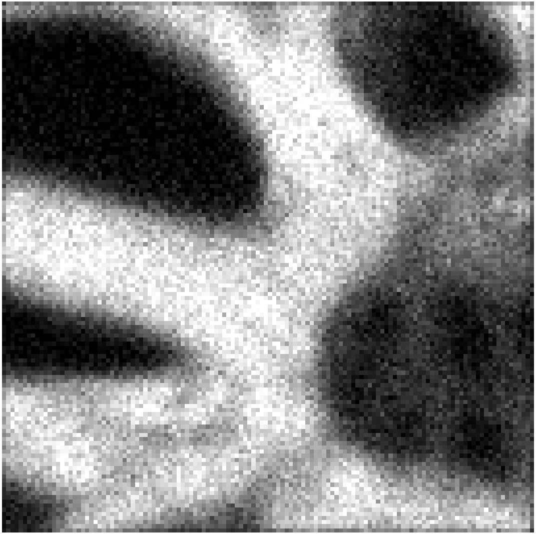
- $P_m = 2N$
- $\boldsymbol{L}_m = \left[(\boldsymbol{\Delta}^{\mathrm{h}})^\top \;\; (\boldsymbol{\Delta}^{\mathrm{v}})^\top \right]^\top$
- $\boldsymbol{\Delta}^{\mathrm{h}} \in \mathbb{R}^{N \times N}$ - horizontal gradient operator
- $\boldsymbol{\Delta}^{\mathrm{v}} \in \mathbb{R}^{N \times N}$ - vertical gradient operator
- $\forall_{\boldsymbol{x} \in \mathbb{R}^N}, r_m(\boldsymbol{L}_m \boldsymbol{x}) = \lambda_m \sum_{n=1}^{N} \left(([\boldsymbol{\Delta}^{\mathrm{h}} \boldsymbol{x}]_n)^2 + ([\boldsymbol{\Delta}^{\mathrm{v}} \boldsymbol{x}]_n)^2 \right)^{1/2}$
- $\lambda_m > 0$

RESULTS



Original image: size 128×128

Results



Original image: size 128×128

Noisy blurred image:
7×7 Gaussian blur with standard deviation 0.5
$\sigma^2 = 50$
MAE = 15.74
SNR = 16.89

RESULTS

Original image: size 128×128

Noisy blurred image:
7×7 Gaussian blur with standard deviation 0.5
$\sigma^2 = 50$
MAE = 15.74
SNR = 16.89

Reconstructed image:
MAE = 5.72
SNR = 25.11

Conclusions

- We have proposed a new EM-based approach, which jointly estimates Poisson and Gaussian noise parameters.
- These algorithms can compute unknown noise parameters for denoising or restoration procedures, which normally are assumed to be known [Benvenuto *et al.* 2008, Luisier *et al.* 2011, Jezierska *et al.* 2012]
- Apart from noise parameters, we can also estimate the rate of bleaching and as a side effect, we can also estimate the original data
- We have proposed a new Proximal-splitting approach adopted to restore data corrupted by Poisson-Gaussian noise

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Future works

- Perform more validations on MACROscope (widefield and confocal)
- Image restoration method with proposed noise model, space varying PSF and more appropriate regularization term than TV (to avoid staircase effect)

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