A PRIMAL-DUAL PROXIMAL SPLITTING APPROACH FOR RESTORING DATA CORRUPTED WITH POISSON-GAUSSIAN NOISE

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INTRODUCTION

State of the art:

✗ **Non-optimality**: Strategies grounded on some approximations of the noise statistics.

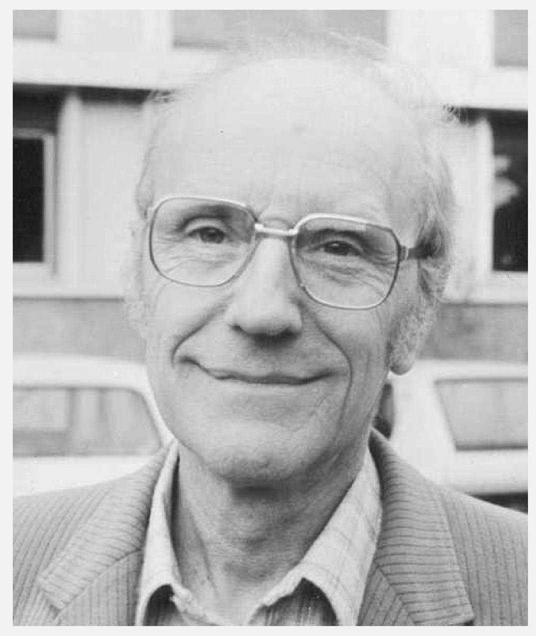
Proposed method:

✓ **New properties**: Poisson-Gaussian neg log likelihood is a convex μ-Lipschitz differentiable function.

✓ **Flexibility**: Restoration algorithm modeling a wide range of prior information, e.g. criteria promoting sparsity in a frame, total-variation and more generally hybrid regularization functions.

✓ **Robustness to numerical errors**: Essential for Poisson-Gaussian model.

TOOL



Jean-Jacques Moreau

1962 Proximity operator

ψ- semi-continuous proper convex function, $\boldsymbol{x} \in \mathbb{R}^N$
$\mathrm{prox}_\psi \colon \mathbb{R}^N \to \mathbb{R}^N$

$$\boxed{\mathrm{prox}_\psi(\boldsymbol{x}) = \min_{\boldsymbol{y}\in\mathbb{R}^N} \psi(\boldsymbol{y}) + \frac{1}{2}\|\boldsymbol{x}-\boldsymbol{y}\|^2}$$

PROBLEM

Degradation model: $\boxed{\boldsymbol{y} = \boldsymbol{z}(\boldsymbol{x}) + \boldsymbol{w}}$

Carl Friedrich Gauss
1809 Normal distribution
$W_i \sim \mathcal{N}(b, \sigma^2)$

Simon Denis Poisson
1827 Poisson distribution
$Z_i(\boldsymbol{x}) \sim \mathcal{P}_\alpha([\boldsymbol{H}\boldsymbol{x}]_i)$

$\boldsymbol{y} \in \mathbb{R}^Q$: $\rightsquigarrow$ observations

$\boldsymbol{z}(\boldsymbol{x}) \in \mathbb{R}^Q$: $\rightsquigarrow$ realization of $\boldsymbol{Z}(\boldsymbol{x}) = (Z_i(\boldsymbol{x}))_{1\le i\le Q}$

$\boldsymbol{w} \in \mathbb{R}^Q$: $\rightsquigarrow$ realization of $\boldsymbol{W} = (W_i)_{1\le i\le Q}$

$\boldsymbol{H} \in [0,+\infty)^{Q\times N}$: $\rightsquigarrow$ linear operator

Problem formulation: $\boxed{\text{Find } \hat{\boldsymbol{x}} \in \arg\min f(\boldsymbol{x})}$

$$\boxed{f(\boldsymbol{x}) = h(\boldsymbol{x}) + r_0(\boldsymbol{x}) + \sum_{m=1}^M r_m(\boldsymbol{L}_m\boldsymbol{x})}$$

- $r_m(\boldsymbol{L}_m\boldsymbol{x})$ - convex regularization term with linear operator $\boldsymbol{L}_m \in \mathbb{R}^{P_m\times N}$
- $r_0(\boldsymbol{x})$ - indicator function of a closed convex subset of $[0,+\infty)^N$
- $h(\boldsymbol{x})$ - for non-negative values defined as $-\log(p_{\boldsymbol{Y}}(\boldsymbol{y};\boldsymbol{x}))$ and which takes a quadratic form on $(-\infty,0]^N$.

Poisson-Gaussian Distribution

$$\boxed{p_{\boldsymbol{Y}}(\boldsymbol{y};\boldsymbol{x}) = \prod_{i=1}^Q \left(\sum_{n=0}^{+\infty} \frac{e^{-\alpha[\boldsymbol{H}\boldsymbol{x}]_i}(\alpha[\boldsymbol{H}\boldsymbol{x}]_i)^n}{n!} \frac{e^{-\frac{1}{2\sigma^2}(y_i-b-n)^2}}{\sqrt{2\pi\sigma^2}}\right)}$$

ALGORITHM

Primal-dual splitting algorithm [Combettes and Pesquet, 2011]

Initialization: $\boldsymbol{x}_0 \in \mathbb{R}^N$, $(\forall m \in \{1,\dots,M\})$ $\boldsymbol{v}_{m,0} \in \mathbb{R}^{P_m}$.

For $k = 0,\dots$

$$\begin{array}{l}
\boldsymbol{y}_{1,k} = \boldsymbol{x}_k - \gamma\left(\nabla h(\boldsymbol{x}_k) + \sum_{m=1}^M \boldsymbol{L}_m^\top \boldsymbol{v}_{m,k}\right) + \boldsymbol{a}_k\\
\boldsymbol{p}_{1,k} = \mathrm{prox}_{\gamma r_0}(\boldsymbol{y}_{1,k})\\
\text{For } m = 1,\dots,M\\
\quad \boldsymbol{y}_{2,m,k} = \boldsymbol{v}_{m,k} + \gamma \boldsymbol{L}_m \boldsymbol{x}_k\\
\quad \boldsymbol{p}_{2,m,k} = \boldsymbol{y}_{2,m,k} - \gamma\,\mathrm{prox}_{\gamma^{-1}r_m}(\gamma^{-1}\boldsymbol{y}_{2,m,k})\\
\quad \boldsymbol{q}_{2,m,k} = \boldsymbol{p}_{2,m,k} + \gamma \boldsymbol{L}_m \boldsymbol{p}_{1,k}\\
\quad \boldsymbol{v}_{m,k+1} = \boldsymbol{v}_{m,k} - \boldsymbol{y}_{2,m,k} + \boldsymbol{q}_{2,m,k}\\
\boldsymbol{q}_{1,k} = \boldsymbol{p}_{1,k} - \gamma\left(\nabla h(\boldsymbol{p}_{1,k}) + \sum_{m=1}^M \boldsymbol{L}_m^\top \boldsymbol{p}_{2,m,k}\right) + \boldsymbol{c}_k\\
\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \boldsymbol{y}_{1,k} + \boldsymbol{q}_{1,k}
\end{array}$$

- $\gamma \in (0,+\infty)$.
- $(\boldsymbol{a}_k)_{k\in\mathbb{N}}$ and $(\boldsymbol{c}_k)_{k\in\mathbb{N}}$ - sequences of elements of $\mathbb{R}^N$ corresponding to possible errors in the computation of the gradient of h.

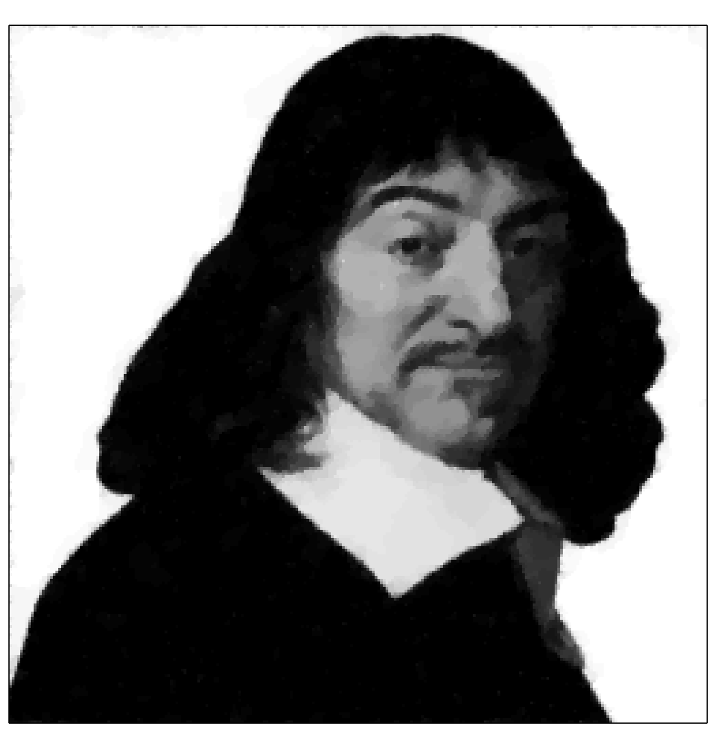
CONVERGENCE

Assumptions:

❶ h is a convex, μ-Lipschitz differentiable function,
❷ f is coercive, i.e. $\lim_{\|\boldsymbol{x}\|\to+\infty} f(\boldsymbol{x}) = +\infty$,
❸ for every $m \in \{1,\dots,M\}$, r_m is finite valued,
❹ $\gamma \in [\epsilon, (1-\epsilon)/\beta]$ where $\epsilon \in (0, 1/(\beta+1))$ and $\beta = \mu + \sqrt{\sum_{m=1}^M \|\boldsymbol{L}_m\|^2}$,
❺ $(\boldsymbol{a}_k)_{k\in\mathbb{N}}$ and $(\boldsymbol{c}_k)_{k\in\mathbb{N}}$ are absolutely summable sequences.

Result: There exists a minimizer $\overline{\boldsymbol{x}}$ of $f(\boldsymbol{x})$ such that the sequences $(\boldsymbol{x}_k)_{k\in\mathbb{N}}$ and $(\boldsymbol{p}_{1,k})_{k\in\mathbb{N}}$ converge to $\overline{\boldsymbol{x}}$.

Noisy image: MAE = 18.98
$(\alpha = 0.4, \sigma^2 = 50)$



Our result: MAE = 3.23
(GAST model: MAE = 3.38)

Noisy blurred image: MAE = 20.48
$(\alpha = 0.4, \sigma^2 = 50)$

Our result: MAE = 3.59
(GAST model: MAE = 3.71)