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# Strong pseudomonotonicity, sharp efficiency and stability for parametric vector equilibria

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**Abstract.** We investigate stability of parametric vector equilibrium problems. We prove Hölder type continuities of solutions. Our results rely on the strong pseudomonotonicity of the bifunctions defining problems. We show that in the case of vector optimization problems the strong pseudomonotonicity introduced in the present paper implies the uniform (with the same constant) sharpness of solutions.

**Key words:** Parametric vector equilibrium problems, Hölder continuity, pseudomonotone bifunctions, sharp solutions.

## 1 Introduction

Let  $X$  and  $Y$  be normed vector spaces and let  $\mathcal{K} \subset Y$  be a closed convex pointed cone in  $Y$  with nonempty interior,  $\text{int}\mathcal{K} \neq \emptyset$ . Let  $C \subset X$  be a subset of  $X$  and let  $F : C \times C \rightarrow Y$  be a mapping (bifunction) satisfying the condition

$$F(x, x) \geq 0 \text{ for } x \in C.$$

Our goal here is to investigate two types of vector equilibrium problems:

- *vector equilibrium problems (WVEP)*: find  $x \in C$  such that

$$F(x, y) \not\leq 0 \text{ for all } y \in C, \quad (1)$$

i.e.,

$$-F(x, y) \notin \text{int}\mathcal{K} \text{ for all } y \in C,$$

- *vector equilibrium problems (VEP)*: find  $x \in C$  such that

$$F(x, y) \not\leq 0 \text{ for all } y \in C, \quad (2)$$

i.e.,

$$-F(x, y) \notin \mathcal{K} \text{ for all } y \in C.$$

By using the efficiency we can reformulate both (*WVEP*) and (*VEP*) in the following way.

Let  $\varphi : C \rightrightarrows Y$  be a multimapping which assigns to each  $x \in C$  the set of efficient points of  $F(x, C)$ , i.e.  $\varphi(x) = E(F(x, C))$ , where for any subset  $A \subset Y$ ,  $y \in E(A)$ , if  $y \in A$  and  $(A - y) \cap (-\mathcal{K}) = \{y\}$ . Then

$$x_0 \in C \text{ solves (VEP)} \Leftrightarrow 0 \in \varphi(x_0)$$

Let  $\psi : C \rightrightarrows Y$  be a multimapping defined as  $\psi(x) = WE(F(x, C))$ , where for any subset  $A \subset Y$ ,  $y \in WE(A)$ , if  $y \in A$  and  $(A - y) \cap (-\text{int}\mathcal{K}) = \{y\}$ . Then

$$x_0 \in C \text{ solves (WVEP)} \Leftrightarrow 0 \in \psi(x_0)$$

Vector equilibrium problems provide a unifying framework for investigating a large variety of problems of variational analysis such as vector variational inequalities, vector optimization problems, variational inequalities, optimization problems. (see e.g. [9, 18]). (*VEP*) generalize vector variational inequalities (*VVI*) introduced in [11].

Let  $f : X \rightarrow 2^{L(X, Y)}$  be a mapping which assigns to an element  $x \in X$  a linear continuous operator acting from  $X$  into  $Y$ . When

$$F(x, y) = \langle f(x), y - x \rangle$$

then (*WVEP*) gives rise to *vector variational inequality (VVI)* : find  $x \in C$  such that

$$\langle f(x), y - x \rangle \not\prec 0 \quad \text{for all } y \in C.$$

For any mapping  $\phi : X \rightarrow Y$ , by letting

$$F(x, y) = \phi(y) - \phi(x)$$

in (*WVEP*) and (*VEP*) we obtain respectively

- *vector optimization problem (WVOP)* : find  $x \in C$  such that

$$\phi(y) - \phi(x) \notin \text{int}\mathcal{K} \quad \text{for all } y \in C,$$

- *vector optimization problem (VOP)* : find  $x \in C$  such that

$$\phi(y) - \phi(x) \notin \mathcal{K} \quad \text{for all } y \in C.$$

Some more relations between (*VVI*) and (*VOP*) are given by Yang and Gog [23].

Our goal here is to investigate Hölder properties of solutions to parametric (*VEP*) and (*WVEP*) problems. We prove our results under the assumption of strong pseudomonotonicity of  $F$ . It is worth mentioning that the definition

of strong pseudomonotonicity adopted here does not imply the uniqueness of solutions to vector equilibrium problems. The strong pseudomonotonicity of real-valued bifunctions  $F$  (with  $Y = \mathbb{R}$ ) has been already used in investigations of Hölder stability of solutions to parametric (scalar) set-valued variational inequalities (VI) in [17], and (scalar) equilibrium problems [3]. For other variants of pseudomonotonicity see [21]. Hölder continuities of solutions to parametric variational inequalities were also investigated in [8, 24] under the strong monotonicity assumption.

The organization of the paper is as follows. In Section 3 we discuss the strong pseudomonotonicity for bifunctions  $F : C \times C \rightarrow Y$ . In Section 4 we prove our main results on Hölder continuity of the solutions to parametric (VEP). In Section 5 we specialize the obtained results for vector optimization problems. In a similar way one can apply the results of Section 4 to vector variational inequalities, variational inequalities, and optimization problems, obtaining in these way Hölder stability results in these class of problems (see also [3, 8, 17, 24]).

## 2 Monotonicity concepts

In this section we introduce basic monotonicity concepts for bifunctions, namely monotonicity, pseudomonotonicity and quasimonotonicity as defined in [4, 5]. These concepts generalize monotonicity, pseudomonotonicity and quasimonotonicity for functions used in investigations of variational inequalities and equilibrium problems e.g. in [7, 10, 12, 19] and in deriving characterizations of generalized convexity e.g. in [15, 16].

For any elements  $x, y \in Y$  we will use the following notations:

$$\begin{array}{ll} x < y \Leftrightarrow y - x \in \text{int}\mathcal{K} & x \leq y \Leftrightarrow y - x \in \mathcal{K} \\ x \not< y \Leftrightarrow y - x \notin \text{int}\mathcal{K} & x \not\leq y \Leftrightarrow y - x \notin \mathcal{K} \\ x \not> y \Leftrightarrow y \not< x & x \not\geq y \Leftrightarrow y \not\leq x. \end{array}$$

We say that  $F$  is:

- *monotone* if for all  $x, y \in C$

$$F(y, x) + F(x, y) \not\geq 0,$$

- *pseudomonotone on  $C$*  if for all  $x, y \in C$

$$F(x, y) \not< 0 \rightarrow F(y, x) \not\geq 0,$$

- *strictly pseudomonotone on  $C$*  if for all  $x, y \in C$

$$F(x, y) \not< 0 \rightarrow F(y, x) < 0,$$

- *quasimonotone on  $C$*  if for all  $x, y \in C$

$$F(x, y) > 0 \rightarrow F(y, x) \leq 0$$

Monotonicity implies pseudomonotonicity which, in turn, implies quasimonotonicity. These concepts and their variants were used in [1, 4, 5, 13] in proving the existence of solutions to vector equilibrium problems (*WVEP*) and (*VEP*).

### 3 Strong pseudomonotonicity

For any element  $z \in Y$  the relation  $z \not\prec \alpha B_Y$ , means that  $z \not\prec \alpha b$  i.e.,  $z - \alpha b \notin \text{int}\mathcal{K}$  for all  $b \in B_Y$ . By the symmetry of balls, the latter condition yields that  $(z + \alpha B_Y) \cap \text{int}\mathcal{K} = \emptyset$  or  $z \cap [\alpha B_Y + \text{int}\mathcal{K}] = \emptyset$ . Analogously, the relation  $z \not\preceq \alpha B_Y$  means that  $z \cap [\alpha B_Y + \mathcal{K}] = \emptyset$ . Moreover, for any sufficiently small  $\varepsilon > 0$

$$z \cap [\alpha B_Y + \text{int}\mathcal{K}] = \emptyset \Leftrightarrow z \cap [(\alpha - \varepsilon)B_Y + \mathcal{K}] = \emptyset. \quad (3)$$

Indeed, if  $z \notin [\alpha B_Y + \text{int}\mathcal{K}]$ , then  $z \notin \mathcal{K}$  and there exists  $0 < \beta \leq \alpha$  such that  $z \cap [\beta B_Y + \mathcal{K}] = \emptyset$ .

We say that  $F$  is:

- *strongly monotone on  $C$*  if there exists  $\alpha > 0$  such that for any pair of distinct points  $x, y \in C$  we have that

$$F(x, y) + F(y, x) \not\prec \alpha \|y - x\|^2 B_Y,$$

- *strongly pseudomonotone* if there exists  $\beta > 0$  such that for any pair of distinct points  $x, y \in C$  we have that

$$F(x, y) \not\prec 0 \rightarrow F(y, x) \not\prec \alpha \|y - x\|^2 B_Y.$$

In view of (3), the strong monotonicity holds iff there exists  $\alpha > 0$  such that for any pair of distinct points  $x, y \in C$

$$F(x, y) + F(y, x) \not\preceq \alpha \|y - x\|^2 B_Y,$$

and the strong pseudomonotonicity holds iff there exists  $\alpha > 0$  such that for any pair of distinct points  $x, y \in C$

$$F(x, y) \not\prec 0 \rightarrow F(y, x) \not\preceq \alpha \|y - x\|^2 B_Y.$$

Moreover, by (3), the relation  $F(y, x) \cap (\alpha B_Y + \text{int}\mathcal{K}) = \emptyset$  implies that  $F(y, x) \notin \mathcal{K}$  which shows that the strong pseudomonotonicity implies that for any  $x, y \in C$ ,  $x \neq y$ ,

$$F(x, y) \not\prec 0 \rightarrow F(y, x) \not\preceq 0.$$

which in turn implies that  $F$  is pseudomonotone.

Clearly, the strong pseudomonotonicity of  $F$  implies that for any small  $\varepsilon > 0$  and any pair of distinct points  $x, y \in C$  we have that

$$F(x, y) \not\leq 0 \rightarrow F(y, x) \not\leq (\alpha - \varepsilon)\|y - x\|^2 B_Y. \quad (4)$$

The converse implication does not hold in general. However, the following holds true.

**Proposition 3.1** *If  $F(y, x) = -F(x, y)$ , then the strong pseudomonotonicity holds if and only if (4) holds.*

**Proof.** Take any distinct  $x, y \in C$  such that  $F(x, y) \not\leq 0$ . By the strong pseudomonotonicity,  $F(y, x) \not\leq 0$  and hence  $F(x, y) \not\leq 0$ .

□

This means, for instance, that in the case of vector variational inequalities and vector optimization problems and variational inequalities the strong pseudomonotonicity is equivalent to (4).

### Example 3.1

1. Consider the bifunction  $F : C \times C \rightarrow Y$  defined as  $F(x, y) = \phi(y) - \phi(x)$ , where  $\phi : X \rightarrow Y$  is a mapping from  $X$  into  $Y$ . Note that  $F(x, y) = -F(y, x)$  and (VEP) reduces to (VOP). By Proposition 3.1,  $F$  is strongly pseudomonotone iff (4) holds, i.e. there exists  $\alpha > 0$  such that for any  $x, y \in C$

$$\phi(y) - \phi(x) \not\leq 0 \rightarrow \phi(y) - \phi(x) \not\leq \alpha\|y - x\|^2 B_Y. \quad (5)$$

Let  $x_0 \in C$  be a solution to (VOP), i.e.,  $\phi(y) - \phi(x_0) \not\leq 0$  for any  $y \in C$ . Then the strong pseudomonotonicity of  $F$  yields that

$$(\phi(y) - \phi(x_0)) \cap (\alpha\|y - x_0\|^2 B_Y - \mathcal{K}) = \emptyset,$$

which means that  $x_0$  is a sharp solution to (VOP) as defined in [6, 14]. Moreover, each solution  $x_0$  is sharp with the same constant  $\alpha > 0$ .

2. Let  $F : C \times C \rightarrow Y$  be given as  $F(x, y) = \langle f(x), y - x \rangle$ , where  $f : X \rightarrow L(X, Y)$  is a mapping from  $X$  into the space of all continuous linear operators from  $X$  into  $Y$ . Then (WVEP) reduces to (VVI) of finding all  $x \in C$  such that  $\langle f(x), y - x \rangle \not\leq 0$  for all  $y \in C$  and  $F$  is strongly pseudomonotone iff there exists  $\alpha > 0$  such that for any  $x, y \in C$

$$\langle f(x), y - x \rangle \not\leq 0 \rightarrow \langle f(y), y - x \rangle \not\leq \alpha\|x - y\|^2 B_Y.$$

3. Let  $F : C \times C \rightarrow R$ . Then (WVEP) reduces to the equilibrium problem (EP) of finding  $x \in C$  such that  $f(x, y) \geq 0$  for all  $y \in C$  and  $F$  is strongly pseudomonotone iff there exists  $\alpha > 0$  such that for all  $x, y \in C$

$$F(x, y) \geq 0 \rightarrow F(y, x) \leq -\alpha \|x - y\|^2,$$

as defined in [3].

4. Let  $F : C \times C \rightarrow R$  be of the form  $F(x, y) = \langle f(x), y - x \rangle$ , where  $f : X \rightarrow X$ . Then (WVEP) reduces to (VI) of finding  $x \in C$  such that  $\langle f(x), y - x \rangle \geq 0$  for all  $y \in C$  and  $F$  is strongly pseudomonotone iff there exists  $\alpha > 0$  such that

$$\langle f(x), y - x \rangle \geq 0 \rightarrow \langle f(y), y - x \rangle \geq \alpha \|x - y\|^2$$

as defined e.g. in [21, 22].

It is easy to prove that in the case of (EP) the strong pseudomonotonicity implies the uniqueness of solutions. In general, the solution set to (VEP) satisfying the strong pseudomonotonicity need not be unique.

## 4 Applications to stability of (VEP)

Let  $U$  be a normed space with the open unit ball  $B_U$ . Consider the parametric vector equilibrium problem (VEP) $_u$  of finding all  $x \in C(u)$  such that

$$F(u, x, y) \not\leq 0 \quad y \in C(u),$$

where  $F : U \times X \times X \rightarrow Y$  is a bifunction and  $C : U \rightrightarrows X$  is a set-valued mapping. In this section we investigate the behaviour of the set-valued mapping  $S : U \rightrightarrows X$  defined as

$$S(u) = \{x \in C(u) : \forall y \in C(u) \quad F(u, x, y) \not\leq 0\}$$

with  $S(u_0) = S_0$ . We recall that the set-valued mapping  $C$  is Hölder of order  $m > 0$  around  $u_0$  if there exist constants  $t_c > 0$ ,  $L_c > 0$  such that  $C(u) \subset C(u_0) + L_c \|u - u_0\| B_Y$  for  $u, u' \in u_0 + t_c B_U$ . Moreover,  $C$  is upper Hölder of order  $m > 0$  at  $u_0$  if there exist constants  $t_c > 0$ ,  $L_c > 0$  such that  $C(u) \subset C(u_0) + L_c \|u - u_0\| B_Y$  for  $u \in u_0 + t_c B_U$ , and  $C$  is lower Hölder of order  $m > 0$  at  $u_0$  if there exist constants  $t_c > 0$ ,  $L_c > 0$  such that  $C(u_0) \subset C(u) + L_c \|u - u_0\| B_Y$  for  $u \in u_0 + t_c B_U$ . If  $C$  is Hölder, or upper Hölder, or lower Hölder of order  $m = 1$ , then  $C$  is called Lipschitz, upper Lipschitz, or lower Lipschitz, respectively.

**Theorem 4.1** Let  $U$ ,  $X$ , and  $Y$  be normed vector spaces and let  $K \subset Y$  be a closed convex pointed cone in  $Y$ . Suppose that  $S(u) \neq \emptyset$  for all  $u$  in some neighbourhood of  $u_0$ . If



(i)  $F$  is Lipschitz around  $u_0 \times X \times X$ ,

(ii)  $C$  is Lipschitz around  $u_0$ ,

(iii) there exists a neighbourhood  $U_0$  of  $u_0$  such that for each  $u \in U_0$  and each  $x \in C(u)$  there is  $x_0 \in S(u)$  such that  $F(u, x_0, x) \in \mathcal{K}$ ,

(iv)  $F(u, \cdot, \cdot)$  are uniformly strongly pseudomonotone on  $C(u)$  for  $u \in U_0$ , i.e., there is  $\beta > 0$  such that for any  $u \in U_0$  and any  $x, x_0 \in C(u)$ ,  $x \neq x_0$ ,

$$F(u, x_0, x) \not\leq 0 \rightarrow F(u, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}$$

then the solution set-valued mapping  $S$  is Hölder of order  $\frac{1}{2}$  around  $u_0$ .

**Proof.** Take any  $u, u' \in U_1 \cap U_0$  and  $x_0 \in S(u)$ . By (ii), there exists  $z \in C(u')$  such that

$$\|x_0 - z\| \leq L_c \|u - u'\|.$$

By (iii), there exists  $z_0 \in S(u')$  such that

$$F(u', z_0, z) \in \mathcal{K}.$$

Again, by (ii), there exists  $x \in C(u)$  such that

$$\|z_0 - x\| \leq L_c \|u - u'\|$$

By Lipschitzness of  $F$ ,

$$\|F(u, x, x_0) - F(u', z_0, z)\| \leq L_f (\|x_0 - z\| + \|x - z_0\| + \|u - u'\|) \leq L_f (2L_c + 1) \|u - u'\|.$$

By this,

$$F(u, x, x_0) \in F(u', z_0, z) + L_f (2L_c + 1) \|u - u'\| B_Y \in L_f (2L_c + 1) \|u - u'\| B_Y + \mathcal{K}. \quad (6)$$

On the other hand, by (iv),

$$F(u, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}. \quad (7)$$

Combining (6) and (7) we obtain

$$\|x - x_0\|^2 \leq \frac{L_f (2L_c + 1)}{\beta} \|u - u'\|$$

and finally, assuming that  $\|u - u'\| < 1$  we get

$$\|x_0 - z_0\| \leq \|x_0 - x\| + \|x - z_0\| \leq \left( L_c + \sqrt{\frac{L_f (2L_c + 1)}{\beta}} \right) \|u - u'\|^{\frac{1}{2}}$$

□

**Theorem 4.2** Let  $U$ ,  $X$ , and  $Y$  be normed vector spaces and let  $\mathcal{K} \subset Y$  be a closed convex pointed cone in  $Y$ . Suppose that  $S(u) \neq \emptyset$  for all  $u$  in some neighbourhood of  $u_0$ . If

- (i)  $F$  is Lipschitz around  $u_0 \times X \times X$
- (ii)  $C$  is Lipschitz around  $u_0$ ,
- (iii) for each  $u \in U_0$  and each  $x \in C(u)$  there is  $x_0 \in S(u)$  such that  $F(u, x_0, x) \in \mathcal{K}$ ,
- (iv)  $F(u_0, \cdot, \cdot)$  is strongly pseudomonotone on  $C$ , i.e., there is  $\beta > 0$  such that for each  $x, x_0 \in C$ ,  $x \neq x_0$ ,

$$F(u_0, x_0, x) \not\leq 0 \rightarrow F(u_0, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}$$

then the solution set-valued mapping  $S$  is lower Hölder of order  $\frac{1}{2}$  around  $u_0$ .

**Proof.** Take any  $u \in U_1 \cap U_0$  and  $x_0 \in S(u_0)$ . By (ii), there exists  $z \in C(u)$  such that

$$\|x_0 - z\| \leq L_c \|u_0 - u\|.$$

By (iii), there exists  $z_0 \in S(u)$  such that

$$F(u, z_0, z) \in \mathcal{K}.$$

Again, by (ii), there exists  $x \in C$  such that

$$\|z_0 - x\| \leq L_c \|u_0 - u\|$$

By Lipschitzness of  $F$ ,

$$\|F(u_0, x, x_0) - F(u, z_0, z)\| \leq L_f (\|x_0 - z\| + \|x - z_0\| + \|u - u_0\|) \leq L_f (2L_c + 1) \|u_0 - u\|.$$

By this,

$$F(u_0, x, x_0) \in F(u, z_0, z) + L_f (2L_c + 1) \|u_0 - u\| B_Y \in L_f (2L_c + 1) \|u_0 - u\| B_Y + \mathcal{K}. \quad (8)$$

On the other hand, by (iv),

$$F(u_0, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}. \quad (9)$$

Combining (8) and (9) we obtain

$$\|x - x_0\|^2 \leq \frac{L_f (2L_c + 1)}{\beta} \|u_0 - u\|$$

and finally

$$\|x_0 - z_0\| \leq \|x_0 - x\| + \|x - z_0\| \leq \left( L_c + \sqrt{\frac{L_f (2L_c + 1)}{\beta}} \right) \|u_0 - u\|^{\frac{1}{2}}$$

□

**Theorem 4.3** Let  $U$ ,  $X$ , and  $Y$  be normed vector spaces and let  $\mathcal{K} \subset Y$  be a closed convex pointed cone in  $Y$ . Suppose that  $S(u) \neq \emptyset$  for all  $u$  in some neighbourhood of  $u_0$ . If

(i)  $F$  is Lipschitz around  $u_0 \times X \times X$

(ii)  $C$  is Lipschitz around  $u_0$ ,

(iii) for each  $x \in C(u_0) = C$  there is  $x_0 \in S_0$  such that  $F(u_0, x_0, x) \in \mathcal{K}$ ,

(iv) there exists a neighbourhood  $U_0$  of  $u_0$  such that all  $F(u, \cdot, \cdot)$  are uniformly strongly pseudomonotone on  $C(u)$  for  $u \in U_0$ , i.e., there is  $\beta > 0$  such that for all  $u \in U_0$  and any  $x, x_0 \in C(u)$ ,  $x \neq x_0$ ,

$$F(u, x_0, x) \not\leq 0 \rightarrow F(u, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}$$

then the solution set-valued mapping  $S$  is upper Hölder of order  $\frac{1}{2}$  around  $u_0$ .

**Proof.** Take any  $u \in U_1 \cap U_0$  and  $x_0 \in S(u)$ . By (ii), there exists  $z \in C$  such that

$$\|x_0 - z\| \leq L_c \|u - u_0\|.$$

By (iii), there exists  $z_0 \in S_0$  such that

$$F(u_0, z_0, z) \in \mathcal{K}.$$

Again, by (ii), there exists  $x \in C(u)$  such that

$$\|z_0 - x\| \leq L_c \|u - u_0\|$$

By Lipschitzness of  $F$ ,

$$\|F(u, x, x_0) - F(u_0, z_0, z)\| \leq L_f (\|x_0 - z\| + \|x - z_0\| + \|u - u_0\|) \leq L_f (2L_c + 1) \|u - u_0\|.$$

By this,

$$F(u, x, x_0) \in F(u_0, z_0, z) + L_f (2L_c + 1) \|u - u_0\| B_Y \in L_f (2L_c + 1) \|u - u_0\| B_Y + \mathcal{K}. \quad (10)$$

On the other hand, by (iv),

$$F(u, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}. \quad (11)$$

Combining (10) and (11) we obtain

$$\|x - x_0\|^2 \leq \frac{L_f (2L_c + 1)}{\beta} \|u - u_0\|$$

and finally

$$\|x_0 - z_0\| \leq \|x_0 - x\| + \|x - z_0\| \leq \left( L_c + \sqrt{\frac{L_f (2L_c + 1)}{\beta}} \right) \|u - u_0\|^{\frac{1}{2}}$$

□

**Remark 4.1** Let us note that in Theorems 4.1 and 4.3 the condition (iv) can be replaced by the weaker condition

(iv)' there exists a neighbourhood  $U_0$  of  $u_0$  such that  $F(u, \cdot, \cdot)$  are uniformly strongly pseudomonotone on  $C(u)$  for  $u \in U_0$ , i.e., there is  $\beta > 0$  such that for all  $u \in U_0$  each  $x_0 \in S(u)$  and all  $x \in C(u)$ ,  $x \neq x_0$ ,

$$F(u, x_0, x) \leq 0 \rightarrow F(u, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}$$

In a similar way the Hölder stability results can be proved for parametric problems  $(WVEP)_u$  of finding  $x \in C(u)$  such that

$$F(u, x, y) \not\leq 0 \quad \text{for all } y \in C(u), \quad (12)$$

For instance Theorem 4.1 takes the following form.

**Theorem 4.4** Let  $U$ ,  $X$ , and  $Y$  be normed vector spaces and let  $\mathcal{K} \subset Y$  be a closed convex pointed cone in  $Y$ . Suppose that  $S(u) \neq \emptyset$  for all  $u$  in some neighbourhood of  $u_0$ . If

- (i)  $F$  is Lipschitz around  $u_0 \times X \times X$ ,
- (ii)  $C$  is Lipschitz around  $u_0$ ,
- (iii) there exists a neighbourhood  $U_0$  of  $u_0$  such that for each  $u \in U_0$  and each  $x \in C(u)$  there is  $x_0 \in S(u)$  such that  $F(u, x_0, x) \in \text{int}\mathcal{K}$ ,
- (iv)  $F(u, \cdot, \cdot)$  is uniformly strongly pseudomonotone on  $C(u)$ ,  $u \in U_0$ , i.e., there is  $\beta > 0$  such that for each  $x, x_0 \in C(u)$

$$F(u, x_0, x) \not\leq 0 \rightarrow F(u, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}$$

then the solution set-valued mapping  $S$  is Hölder of order  $\frac{1}{2}$  around  $u_0$ .

Theorems 4.2 and 4.3 can be rephrased in a similar way.

## 5 Special case: vector optimization problems

As pointed out in Introduction, vector equilibrium problems (VEP) with  $F(x, y) = \phi(y) - \phi(x)$ , where  $\phi : X \rightarrow Y$  is a mapping from  $X$  into  $Y$ , reduce to vector optimization problems (VOP) of finding all  $x \in C$  such that  $\phi(x) \in E(\phi(C))$ . Now we apply the results of the previous section to parametric vector optimization  $(VOP)_u$ ,

$$(VOP)_u \quad \mathcal{K} - \min_{x \in C(u)} \phi(u, x)$$

of finding  $S(u) = \{x \in C(u) \mid \phi(u, x) \in E(\phi(u, C(u)))\}$ , with  $S(u_0) = S_0$ .

In the case of vector optimization problems the condition (iii) of Theorem 4.1 is rephrased as follows: there exists a neighbourhood  $U_0$  of  $u_0$  such that for  $u \in U_0$  and each  $x \in C(u)$  there exists  $x_0 \in S(u)$  such that  $\phi(u, x) - \phi(u, x_0) \in \mathcal{K}$ . In other words, there exists a neighbourhood  $U_0$  of  $u_0$  such that the domination property (DP) holds for  $(VOP)_u$ ,  $u \in U_0$  (see [20]).

As shown in Example 3.1.1, in the case of vector optimization problems  $(VOP)$ , the strong pseudomonotonicity implies that all the solutions to  $(VOP)$  are sharp as defined in [6, 14] with the same constant  $\beta$ .

**Theorem 5.1** *Let  $U$ ,  $X$ , and  $Y$  be normed vector spaces and let  $\mathcal{K} \subset Y$  be a closed convex pointed cone in  $Y$ . Suppose that  $S(u) \neq \emptyset$  for all  $u$  in some neighbourhood of  $u_0$ . If*

- (i)  $\phi$  is Lipschitz around  $u_0 \times X$ ,
- (ii)  $C$  is Lipschitz around  $u_0$ ,
- (iii) there exists a neighbourhood  $U_0$  of  $u_0$  such that (DP) holds for all  $(VOP)_u$  with  $u \in U_0$ ,
- (iv) the solution sets  $S(u)$ ,  $u \in U_0$ , are uniformly sharp, i.e., there is  $\beta > 0$  such that for each  $x \in C(u)$  and  $x_0 \in S(u)$ ,  $x \neq x_0$ ,

$$(\phi(u, x) - \phi(u, x_0)) \cap (\beta \|x - x_0\|^2 B_Y - \mathcal{K}) = \emptyset,$$

then the solution set-valued mapping  $S$  is Hölder of order  $\frac{1}{2}$  around  $u_0$ .

In a similar way Theorem 4.2 and Theorem 4.3 can be rewritten for  $(VOP)_u$ . In [6] weak sharp solutions to vector optimization are defined and the Hölder type continuities of weak sharp solutions to parametric problems are investigated. The generalization of this approach to  $(VEP)$  will be given elsewhere.

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