Partial differential equations

Computer Modeling

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Definicja

Partial differential equation for a function $u(x, y, \dots)$ is a relationship between u and its partial derivatives u_x, $u_y, \dots$, u_{xx}, u_{xy}, $u_{yy}, \dots$ and can be written as:

$$F(x, y, \dots, u, u_x, u_y, \dots, u_{xx}, u_{xy}, u_{yy}, \dots) = 0, \qquad (\star)$$

where F is some function, $x, y, \dots$ are independent variables and $u(x, y, \dots)$ is called a *dependent variable*

The *first-* and the *second-order* partial differential equation with two independent variables x and y, can be written as follows:

$$F(x, y, u, u_x, u_y) = 0, \quad F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \qquad (1)$$

and so on for higher-order equations.

Examples:

- 1st-order equations: $xu_x + yu_y = 0$, $xu_x + yu_y = x^2$, $uu_x + u_t = u$, $u_x^2 + u_y^2 = 1$.
- 2nd-order equations: $u_{xx} + 2u_{xy} + u_{yy} = 0$, $u_{xx} + u_{yy} = 0$, $u_{tt} - c^2 u_{xx} = f(x, t)$.
- 3rd-order and 4th-order equations: $u_t + uu_x + u_{xxx} = 0$, $u_{tt} + u_{xxxx} = 0$.

Linear operator

It is possible to write a PDE in the operator form

$$L_{\mathbf{x}} u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, \dots). \tag{2}$$

$L_{\mathbf{x}}$ is called *linear operator* if it satisfies the property

$$L_{\mathbf{x}}(au + bv) = aL_{\mathbf{x}}u + bL_{\mathbf{x}}v \tag{3}$$

for any two functions u and v and for any two constants a and b.
Equation (2) is called *linear* if $L_{\mathbf{x}}$ is a linear operator.
If $L_{\mathbf{x}}$ is not linear, then (2) is called *nonlinear equation*
If $f(\mathbf{x}) \neq 0$, equation (2) is called *nonhomogeneous*.
If $f(\mathbf{x}) = 0$, equation (2) is called *homogeneous* equation.

Examples:

- equations: $xu_x + yu_y = 0$, $u_{xx} + 2u_{xy} + u_{yy} = 0$, $u_{xx} + u_{yy} = 0$, and $u_{tt} + u_{xxxx} = 0$ are linear homogeneous equations
- equations: $xu_x + yu_y = x^2 + y^2$, $u_{tt} - c^2 u_{xx} = f(x, t)$ are linear inhomogeneous equations.
- equations $uu_x + u_t = u$, $u_x^2 + u_y^2 = 1$, and $u_t + uu_x + u_{xxx} = 0$ are examples of nonlinear equations

Classical and weak solution of PDE

$$F(x, y, \dots, u, u_x, u_y, \dots, u_{xx}, u_{xy}, u_{yy}, \dots) = 0, \qquad (\star)$$

A *classical solution* (or simply a *solution*) of $(\star)$ is a function

$$u = u(x, y, \dots), \quad x, y, \dots \in D$$

which is continuously differentiable such that all its partial derivatives involved in the equation exist and satisfy $(\star)$ identically.

However, this notion of classical solution can be extended by relaxing the requirement that u is continuously *differentiable* over D. The solution $u = u(x, y, \dots)$ is called a *weak* (or *generalized*) *solution* of $(\star)$ if u or its partial derivatives are discontinuous in some (finite number of) points in D.

General solution

To introduce the idea of a *general solution* of a PDE, we solve a simple equation for $u = u(x, y)$

$$u_{xy} = 0. \tag{4}$$

Integrating (4) with respect to x (keeping y fixed), we obtain

$$u_y = h(y). \tag{5}$$

where $h(y)$ is an arbitrary function of y. We then integrate it with respect to y to find

$$u(x, y) = \int h(y)dy + f(x), \tag{6}$$

where $f(x)$ is an arbitrary function. Or, equivalently

$$u(x, y) = f(x) + g(y). \tag{7}$$

where $f(x)$ and $g(y)$ are arbitrary functions. The solution (7) is called the *general solution* of the second-order equation (4)

Usually, the general solution of a PDE is an expression that involves arbitrary functions (in contrast to the general solution of an ordinary differential equation which involves arbitrary constants). Further, a simple equation $u_{xy} = 0$ has infinitely many solutions. This can be illustrated by considering the problem of construction of PDEs from given arbitrary functions. For example, if

$$u(x,t) = f(x - ct) + g(x + ct), \tag{8}$$

where f and g are arbitrary functions of $(x - ct)$ and $(x + ct)$, respectively, then

$$\begin{aligned} u_{xx} &= f''(x - ct) + g''(x + ct), \\ u_{tt} &= c^2 f''(x - ct) + c^2 g''(x + ct) = c^2 u_{xx}. \end{aligned} \tag{9}$$

Thus, we obtain the second-order linear equation, called the *wave equation*,

$$u_{tt} - c^2 u_{xx} = 0. \tag{10}$$

We see that the function $u(x,t) = f(x - ct) + g(x + ct)$ satisfies (10) irrespective of the functional forms of $f(x - ct)$ and $g(x + ct)$, provided f and g are at least twice differentiable functions. Thus, the general solution of equation (10) is given by $u(x,t) = f(x - ct) + g(x + ct)$ which contains arbitrary functions.

Initial and boundary conditions

In almost all cases, the general solution of a partial differential equation is of little use since it has to satisfy other supplementary conditions, usually called *initial* or *boundary conditions*.

The general solution of a linear partial differential equations contains arbitrary functions.

This means that there are infinitely many solutions and only by specifying the *initial* and/or *boundary conditions* can we determine a specific solution of interest.

Usually, both initial and boundary conditions arise from the physics of the problem.

Cauchy (or initial) conditions

In the case of partial differential equations in which one of the independent variables is the time t, an initial condition specifies the physical state of the dependent variable $u(\mathbf{x}, t)$ at a particular time $t = t_0$ or $t = 0$. Often $u(\mathbf{x}, 0)$ and/or $u_t(\mathbf{x}, 0)$ are specified to determine the function $u(\mathbf{x}, t)$ at later times. Such conditions are called the *Cauchy* or *initial conditions*. It can be shown that these conditions are necessary and sufficient for the existence of a unique solution. The problem of finding the solution of the initial-value problem with prescribed Cauchy data on the line $t = 0$ is called the *Cauchy problem* or the *initial-value problem*.

Boundary conditions

In each physical problem, the governing equation is to be solved within a given domain D of space with prescribed values of the dependent variable $u(\mathbf{x}, t)$ given on the boundary ∂D of D.
There are three important types of boundary conditions which arise frequently in formulating physical problems:

- *Dirichlet conditions* where the solution u is prescribed on the boundary ∂D of the domain D. The problem of finding the solution of a given equation $L_x u(\mathbf{x}) = 0$ inside D with prescribed values of u on ∂D is called the *Dirichlet boundary-value problem*;
- *Neumann conditions*, where valus of normal derivative $\dfrac{\partial u}{\partial n}$ of the solution on the boundary ∂D are specified. In this case, the problem is called the *Neumann boundary-value problem*;
- *Robin conditions*, where $\left(\dfrac{\partial u}{\partial n} + au\right)$ is specified on ∂D. The corresponding problem is called the *Robin boundary-value problem*.

Well-posed PDE problem

A problem described by a partial differential equation in a given domain with a set of initial and/or boundary conditions is said to be *well-posed* (or *properly posed*) provided the following criteria are satisfied:

existence: There exists at least one solution of the problem.
uniqueness: There is at most one solution.
stability: The solution must be stable in the sense that it depends continuously on the data. In other words, a small change in the given data must produce a small change in the solution.

The stability criterion is essential for physical problems. A mathematical problem is usually considered physically realistic if a small change in given data produces correspondingly a small change in the solution.

According to the Cauchy-Kowalewski theorem, the solution of an analytic Cauchy problem for partial differential equation exists and is unique locally. However, a Cauchy problem for Laplace's equation is not always *well-posed*. A famous example of a *non-well-posed* (or *ill-posed*) problem was first given by Hadamard.
The following Hadamar's example deals with Cauchy's initial-value problem for the Laplace equation

$$\Delta u = \nabla^2 u \equiv u_{xx} + u_{tt} = 0, \quad 0 < t < \infty, \quad x \in \mathbb{R} \tag{11}$$

with the Cauchy condition

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = \left(\frac{1}{n}\right) \sin nx, \tag{12}$$

where n is an integer representing the wavenumber. These condition tend to zero uniformly as $n \to \infty$. It is easily to verified that the unique solution of this problem is given by:

$$u(x,t) = \left(\frac{1}{n^2}\right) \sinh nt \sin nx \tag{13}$$

As $n \to \infty$, this solution does not tend to the solution $u = 0$. In fact, solution (13) represents oscillations in x with unbounded amplitude $n^{-2} \sinh nt$ which tends to infinity as $n \to \infty$. In other words, although the data change by an arbitrary small amount, the change in the solution is infinitely large. Thus, the problem is certainly not well-posed, that is, the solution does not depend continuously on the initial data.

Some Important Classical Linear Model Equations

We start with a special type of second-order linear partial differential equations for the following reasons:

- second-order equations arise more frequently in a wide variety of applications;
- their mathematical treatment is simpler and easier to understand than the first-order equations in general;
- in almost all physical phenomena, the dependent variable $u = u(x, y, z, t)$ is a function of three space variable and time variable t.

The wave equation

describes the propagation of a wave (or disturbance).

$$u_{tt} - c^2 \Delta u = 0 \qquad (14)$$

where

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and c is a constant. The wave equation arises in a wide variety of physical problems:

- vibrating string, membrane,
- longitudinal vibrations of an elastic rod or beam,
- shallow water waves,
- acoustic problems for the velocity potential for a fluid flow through which sound can be transmitted,
- transmission of electric signals along a cable,
- both electric and magnetic fields in the absence of charge and dielectric.

The heat or diffusion equation

describes the diffusion of thermal energy in a homogeneous medium.

$$u_t - \kappa \Delta u = 0 \quad (15)$$

where κ is the constant of diffusivity.

- It can be used to model the flow of a quantity, such as heat, or a concentration of particles.
- It is also used as a model equation for growth and diffusion, in general, and growth of a solid tumor, in particular.
- It describes also the diffusion of vorticity from a vortex sheet.

The Laplace equation

$$\Delta u = 0 \tag{16}$$

This equation is used to describe:

- elecrostatic potential in the absence of charges,
- gravitational potential in the absence of mass,
- equilibrium displacement of an elastic membrane,
- velocity potential for an incompressible fluid flow,
- temperature in a steady-state heat conduction problem

and many other physical phenomena.

The Poisson equation and the Helmholtz equation

$$\Delta u = f(x, y, z) \qquad (17)$$

where $f(x, y, z)$ is a given function describing a source or sink. This is an inhomogeneous Laplace equation, and hence, the Poisson equation is used to study all phenomena described by the Laplace equation in the presence of external sources or sinks.

$$\Delta u + \lambda u = 0 \qquad (18)$$

where λ is a constant.
This is a time-independent wave equation with λ as a separation constant.
In particular, its solution in acoustics represents an acoustic radiation potential.

The telegraph equation

$$u_{tt} - c^2 u_{xx} + a u_t + b u = 0 \tag{19}$$

where a, b, and c are constants.
This equation arises in the study of propagation of electrical signals in a cable of transmission line. Both current I and voltage V satisfy en equation of the form (19)
This equation also arises in the propagation of pressure wave in the study of pulsatile blood flow in arteries and in one-dimentional random motion of bugs along a hedge.

The Klein-Gordon (KG) equation

Klein (1927) and Gordon (1926) derived a relativistic equation for a charged particle in an electromagnetic field. It is of conservative dispersive type and played an important role in our understanding of the elementary particles.

$$\Box\psi + \left(\frac{mc^2}{h}\right)\psi = 0, \quad \text{where} \quad \Box \equiv \frac{\partial^2}{\partial t^2} - c^2\Delta \qquad (20)$$

is the d'Alembertian operator, $h(= 2\pi\hbar)$ is the Planck constant, and m is a constant mass of the particle.
This equation is also used to describe dispersive wave phenomena in general.

The time-independent Schrödinger equation in quantum mechanics

$$\left(\frac{\hbar^2}{2m}\right)\Delta\psi + (E - V)\psi = 0 \tag{21}$$

where $h(= 2\pi\hbar)$ is the Planck constant, m is the mass of the particle whose wave function is $\psi(x, y, z, t)$, E is a constant, and V is the potential energy.
If $V = 0$, the Schrödinger equation reduces to the Helmholz equation.

The linear Korteweg-de Vries (KdV) equation and the linear Boussinesq equation

$$u_t + \alpha u_x + \beta u_{xxx} = 0 \qquad (22)$$

where α and β are constants. This equation describes the propagation of linear, long, water waves and of plasma waves in a dispersive medium.

$$u_{tt} + \alpha^2 \Delta u + \beta^2 \Delta u = 0 \qquad (23)$$

where α and β are constants. This equation arises in elasticity for longitudinal waves in bars, long water waves, and plasma waves.

The biharmonic wave equation

$$u_{tt} + c^2\nabla^4 u = 0 \tag{24}$$

where c is a constant.

In elasticity, the displacement of a thin elastic plate by small vibrations satisfies this equation. When u is independent of time t, the biharmonic wave equation reduces to what is called the biharmonic equation

$$\nabla^4 u = 0 \tag{25}$$

This describes the equilibrium equation for the distribution of stresses in an elastic medium satisfied by Airy's stress function $u(x, y, z)$. In fluid dynamics, this equation is satisfied by the stream function $\psi(x, y, z)$ in a viscous fluid flow.

The electromagnetic wave equations

for the electric field E and the polarization P are

$$\mathcal{E}_0(E_{tt} - c_0^2 E_{xx}) + P_{tt} = 0, \tag{26}$$

$$(P_{tt} + \omega_0^2 P) - \mathcal{E}_0 \omega_p^2 E = 0, \tag{27}$$

where $\mathcal{E}_0$ is a permittivity (or dielectric constant) of free space, ω_0 is a natural frequency of the oscillator, c_0 is the speed of light in a vacuum, and ω_p is the plasma frequency.

Second-Order Equations

Definition of Second-Order PDE

The general second-order linear partial differential equation in two independent variables x, y is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \qquad (\star)$$

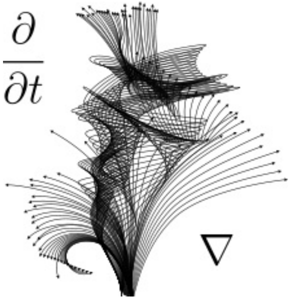
where A, B, C, D, E, F, and G are given functions of x and y or constants.

The classification of second-order equations is based upon the possibility of reducing equation $(\star)$ by a coordinates transformation to a canonical or standard form at a point. Due to this classification we consider the following three types of PDE:

- hyperbolic equation
- parabolic equation
- elliptic equation

Method of characteristics

The method of characteristics is a technique for solving partial differential equations. The method is to reduce a partial differential equation to a family of ordinary differential equations along which the solution can be integrated from some initial data given on a suitable hypersurface.



For a second-order PDE the method of characteristics discovers surfaces (called characteristic surfaces or just characteristics) along which the PDE becomes an ordinary differential equation (ODE). Once the ODE is found, it can be solved along the characteristic surfaces and transformed into a solution for the original PDE.

Method of characteristics

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad (\star)$$

We consider the tranformation from x, y, to ξ, η defined by

$$\xi = \phi(x, y), \quad \eta = \psi(x, y) \qquad (28)$$

where ϕ and ψ are twice continuously differentiable and the Jacobian $J(x, y) = \phi_x\psi_y - \psi_x\phi_y$ is nonzero in a domain of interest so that x, y can be determined uniquely form the system (28). Then, by the chain rule,

$$u_x = u_\xi\xi_x + u_\eta\eta_x, \quad u_y = u_\xi\xi_y + u_\eta\eta_y,$$

$$\begin{aligned}
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}, \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}, \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy},
\end{aligned}$$

Substituting these results in equation $(\star)$ gives

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*, \qquad (\star\star)$$

where

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, & D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y, \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y, & E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y, \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2, & F^* &= F \text{ and } G^* = G.
\end{aligned}$$

Now, the problem is to determine ξ and η so that the equation $(\star\star)$ takes the simplest possible form. We choose ξ and η such that $A^* = C^* = 0$ and $B^* \neq 0$. Or, more explicitly,

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \tag{29}$$

$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \tag{30}$$

These two equations can be combined into a single quadratic equation for $\zeta = \xi$ or η

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0 \tag{31}$$

We consider level curves $\xi = \phi(x, y) =$ constant $= C_1$ and $\eta = \psi(x, y) =$ constant $= C_2$. On these curves

$$d\xi = \xi_x dx + \xi_y dy = 0, \;\; d\eta = \eta_x dx + \eta_y dy = 0,$$

that is, the slopes of these curves are given by

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}, \quad \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}. \tag{32}$$

Thus, the slopes of both level curves are the roots of the same quadratic equation which is obtained from (31) as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \tag{33}$$

and the roots of this equation are given by

$$\frac{dy}{dx} = \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}). \tag{34}$$

These equations are known as the *characteristic equations* for $(\star)$, and their solutions are called the *characteristic curves* or simply *characteristics* of $(\star)$. The solution of the two ordinary differential equations (34) defines two distinct families of characteristics $\phi(x, y) = C_1$ and $\psi(x, y) = C_2$.

Classification of PDE

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*, \qquad (\star\star)$$

Case 1. Equations for which $B^2 - 4AC > 0$ are called *hyperbolic*.
In this case we get two real and distinct families of characteristics $\phi(x,y) = C_1$ and $\psi(x,y) = C_2$, where C_1 and C_2 are constants of integration of the equations $d\xi = 0$ and $d\eta = 0$. Since $A^* = C^* = 0$, and $B^* \neq 0$, and dividing by B^*, equation $(\star\star)$ reduces to the form

$$u_{\xi\eta} = -\frac{1}{B^*}(D^* u_\xi + E^* u_\eta + F^* u - G^*) = H_1 \ (say) \qquad (35)$$

If the new independent variables $\alpha = \xi - \eta$, $\beta = \xi + \eta$ are introduced, then

$$u_\xi = u_\alpha \alpha_\xi + u_\beta \beta_\xi = u_\alpha + u_\beta, \quad u_\eta = u_\alpha \alpha_\eta + u_\beta \beta_\eta = u_\alpha - u_\beta,$$
$$(u_\eta)_\xi = (u_\eta)_\alpha \alpha_\xi + (u_\eta)_\beta \beta_\xi = (u_\alpha - u_\beta)_\alpha \cdot 1 + (u_\alpha - u_\beta)_\beta \cdot 1 = u_{\alpha\alpha} - u_{\beta\beta}$$

Consequently, equation (35) becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = H_1(\alpha, \beta, u, u_\alpha, u_\beta) \ \text{(say)}, \qquad (36)$$

and is called the *canonical form of the hyperbolic equation*.

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*, \qquad (\star\star)$$

Case 2. Equations for which $B^2 - 4AC = 0$ are called parabolic.
There is only one family of real characteristics whose slope is given by

$$\frac{dy}{dx} = \frac{B}{2A} \qquad (37)$$

Integrating this equation gives $\xi = \phi(x, y) = \text{const.}$ (or $\eta = \psi(x, y) = \text{const.}$).
Since $B^2 = 4AC$ and $A^* = 0$, we obtain

$$0 = A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = (\sqrt{A}\xi_x + \sqrt{C}\xi_y)^2.$$

It then follows that

$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 2(\sqrt{A}\xi_x + \sqrt{C}\xi_y)(\sqrt{A}\eta_x + \sqrt{C}\eta_y) = 0$$

for an arbitrary value of η which is independent of ξ.
Dividing $(\star\star)$ by $C^* \neq 0$ yields

$$u_{\eta\eta} = H_2(\xi, \eta, u, u_\xi, u_\eta) \qquad (38)$$

This is known as the *canonical form of the parabolic equation*. On the other hand, if we choose $\eta = \psi(x, y) = \text{constant}$ as the integral of (37), equation $(\star\star)$ assume the form

$$u_{\xi\xi} = H_2^*(\xi, \eta, u, u_\xi, u_\eta) \qquad (39)$$

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*, \qquad (\star\star)$$

Case 3. Equations for which $B^2 - 4AC < 0$ are called elliptic. In this case, equation

$$\frac{dy}{dx} = \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}). \qquad (40)$$

has no real solutions. So there are two families of complex characteristics. Since roots ξ, η of (40) are complex conjugates of each other, we introduce the new variables as $\alpha = \frac{1}{2}(\xi + \eta)$, $\beta = \frac{1}{2i}(\xi - \eta)$, so that $\xi = \alpha + i\beta$ and $\eta = \alpha - i\beta$. We transform $(\star\star)$ into the form

$$A^{**} u_{\alpha\alpha} + B^{**} u_{\alpha\beta} + C^{**} u_{\beta\beta} = H_4(\alpha, \beta, u, u_\alpha, u_\beta) \qquad (41)$$

where the coefficients of this equation assume the same form as the coefficients of $(\star)$. It can be easy to verified that $A^* = 0$ and $C^* = 0$ take the form

$$A^{**} - C^{**} \pm iB^{**} = 0 \;\Leftrightarrow\; A^{**} = C^{**} \text{ and } B^{**} = 0.$$

Thus, dividing by A^{**}, equation (41) reduces to the form

$$u_{\alpha\alpha} + u_{\beta\beta} = H_5(\alpha, \beta, u, u_\alpha, u_\beta) \qquad (42)$$

which is called the *canonical form of the elliptic equation*.

In summary

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad (\star)$$

We state that the equation $(\star)$ is called *hyperbolic*, *parabolic*, or *elliptic* at a point (x_0, y_0) accordingly as

$$B^2(x_0, y_0) - 4A(x_0, y_0)C(x_0, y_0) >=< 0. \qquad (43)$$

If it is true at all points in a given domain, then the equation is said to be *hyperbolic*, *parabolic*, or *elliptic* in that domain.

Finally, it has been shown above that, for the case of two independent variables, a transformation can always be found to transform the given equation to the canonical form. However, in the case of several independent variables, in general, it is not possible to find such a transformation

These three types of partial differential equations arise in many areas of mathematical and physical sciences. Usually

- the boundary value problems are associated with elliptic equations
- the initial-value problems arise in connection with hyperbolic and parabolic equations.

Example

The equation $xu_{xx} + u_{yy} = x^2$ is elliptic for $x > 0$, parabolic for $x = 0$, and hyperbolic for $x < 0$. Indeed, since $A = x$, $B = 0$, $C = 1$, then

$$B^2 - 4AC = -4x \begin{cases} < 0 & \text{for } x > 0, \\ = 0 & \text{for } x = 0, \\ > 0 & \text{for } x < 0. \end{cases}$$

The characteristics equations are

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \pm \frac{1}{\sqrt{-x}} \qquad (44)$$

Hence $y = \pm 2\sqrt{-x} + c = \text{const.}$, or

$$\xi = y + 2\sqrt{-x} = \text{constant}, \qquad (45)$$
$$\eta = y - 2\sqrt{-x} = \text{constant}. \qquad (46)$$

These represents two branches of the parabolas $(y - c)^2 = -4x$ where c is a constant.

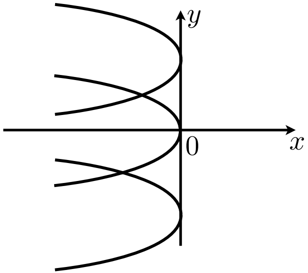


Fig. 1 Characteristics are parabolas for $x < 0$.

Example continued

For $x < 0$, we use the transformations $\xi = y + 2\sqrt{-x}$, $\eta = y - 2\sqrt{-x}$ to reduce the equation $xu_{xx} + u_{yy} = x^2$ to the canonical form. We find

$$\xi_x = -\frac{1}{\sqrt{-x}},\ \xi_y = 1,\ \xi_{xx} = -\frac{1}{2}\frac{1}{(-x)^{3/2}},\ \xi_{yy} = 0,\ \eta_x = +\frac{1}{\sqrt{-x}},\ \eta_y = 1,$$
$$\eta_{xx} = \frac{1}{2}\frac{1}{(-x)^{3/2}},\ \eta_{yy} = 0,\ (\xi - \eta) = 4\sqrt{-x}, \quad (\xi - \eta)^4 = (16x)^2.$$

Consequently, the equation $xu_{xx} + u_{yy} = x^2$ reduces to the form

$$x(u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}) + (u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}) = x^2.$$

$$x\left[u_{\xi\xi}\left(-\frac{1}{x}\right) + 2u_{\xi\eta}\left(\frac{1}{x}\right) - u_{\eta\eta}\left(\frac{1}{x}\right) - \frac{1}{2}\frac{1}{(-x)^{3/2}}u_\xi + \frac{1}{2}\frac{1}{(-x)^{3/2}}u_\eta\right]$$
$$+[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}] = x^2.$$

$$4u_{\xi\eta} + \frac{1}{2}\frac{1}{\sqrt{-x}}(u_\xi - u_\eta) = \frac{1}{16^2}(\xi - \eta)^4 \Rightarrow u_{\xi\eta} = \frac{1}{4}\left(\frac{1}{(16)^2}(\xi - \eta)^4 - \frac{1}{2}\frac{1}{(\xi - \eta)}(u_\xi - u_\eta)\right).$$

This is the first canonical form.

Example continued

For $x > 0$, we use the transformations $\xi = y + 2i\sqrt{x}$, $\eta = y - 2i\sqrt{x}$ so that $\alpha = \frac{1}{2}(\xi + \eta) = y$, $\beta = \frac{1}{2i}(\xi - \eta) = 2\sqrt{x}$. Clearly,

$$\alpha_x = 0, \alpha_y = 1, \alpha_{xx} = 0, \alpha_{yy} = 0, \alpha_{xy} = 0, \beta_x = \frac{1}{\sqrt{x}}, \beta_y = 0, \beta_{xx} = -\frac{1}{2}\frac{1}{x^{3/2}}, \beta_{yy} = 0.$$

So, equation $xu_{xx} + u_{yy} = x^2$ reduces to the canonical form

$$\begin{aligned} &x(u_{\alpha\alpha}\alpha_x^2 + 2u_{\alpha\beta}\alpha_x\beta_x + u_{\beta\beta}\beta_x^2 + u_\alpha\alpha_{xx} + u_\beta\beta_{xx}) \\ &+(u_{\alpha\alpha}\alpha_y^2 + 2u_{\alpha\beta}\alpha_y\beta_y + u_{\beta\beta}\beta_y^2 + u_\alpha\alpha_{yy} + u_\beta\beta_{yy}) = \left(\frac{\beta}{2}\right)^4, \end{aligned}$$

$$u_{\alpha\alpha} + u_{\beta\beta} - \frac{1}{2}\frac{1}{\sqrt{x}}u_\beta = \left(\frac{\beta}{2}\right)^4, \quad \text{thus} \quad u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{\beta}u_\beta + \left(\frac{\beta}{2}\right)^4$$

This is the desired canonical form of the elliptic equation.

Example continued

Finally, for the parabolic case $(x = 0)$, equation $xu_{xx} + u_{yy} = x^2$ reduces to the canonical form

$$u_{yy} = 0.$$

In this case, the characteristic determined from $\frac{dx}{dy} = 0$ is $x = 0$. That is, the $y-$axis is the characteristic curve, and it represents a curve across which a transition from hyperbolic to elliptic form takes place.

The Method of Separation of Variables

The Method of Separation of Variables is widely used to solve initial boundary-value problems involving linear partial differential equations.
Let us assume that the dependent variable $u(x, y)$ is expressed in the separable form

$$u(x, y) = X(x)Y(y), \tag{47}$$

where X and Y are functions of x and y, respectively.
Then, in many cases, the partial differential equation may be reduced to two ordinary differential equations for X and Y.

Example: *Transverse Vibration of a String*

We consider the one dimensional linear wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \ell, \ t > 0, \tag{$\star$}$$

where $c^2 = T^*/\rho$, T^* is a constant tension, and ρ constant line density of the string. The initial and boundary conditions are

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leqslant x \leqslant \ell, \tag{48}$$

$$u(0,t) = 0 = u(\ell,t), \quad t > 0, \tag{49}$$

where f and g are the initial displacements and initial velocity, respectively.
According to the method of separation of variables, we assume a separable solution of the form

$$u(x,t) = X(x)T(t) \neq 0, \tag{50}$$

where X is a function of x only, and T is a function of t only.
Substituting this solution in equation ($\star$) yields

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2} \tag{$\star\star$}$$

Example continued

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} \qquad (\star\star)$$

Since, the left hand side of this equation is a function of x only and the right-hand side is a function of t only, it follows that $(\star\star)$ can be true only if both sides are equal to some constat value. We then write

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = \lambda \implies \frac{d^2X}{dx^2} = \lambda X, \quad \frac{d^2T}{dt^2} = \lambda c^2 T \qquad (51)$$

where λ is an arbitrary separation constant. We solve this pair of equations by using the boundary conditions $u(0,t) = 0 = u(\ell,t)$, $t > 0$:

$$u(0,t) = X(0)T(t) = 0, \text{ for } t > 0, \qquad (52)$$

$$u(\ell,t) = X(\ell)T(t) = 0, \text{ for } t > 0. \qquad (53)$$

Hence, we take $T(t) \neq 0$ to obtain

$$X(0) = 0 = X(\ell). \qquad (54)$$

Example continued

$$\frac{d^2X}{dx^2} = \lambda X, \quad X(0) = 0 = X(\ell) \tag{$\star\star\star$}$$

There are three possible cases: (i) $\lambda > 0$, (ii) $\lambda = 0$, (iii) $\lambda < 0$.

- For case (i), $\lambda = \alpha^2 > 0$. The solution of $(\star\star\star)$ is

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}, \tag{55}$$

  which together with the boundary condition provides to $A = B = 0$. This leads to a trivial solution $u(x, t) = 0$.

- For case (ii), $\lambda = 0$. The solution of $(\star\star\star)$ is

$$X(x) = Ax + B \tag{56}$$

  Then, we use the boundary condition to obtain $A = B = 0$. This also leads to the trivial solution $u(x, t) = 0$.

Example continued

$$\frac{d^2X}{dx^2} = \lambda X, \quad X(0) = 0 = X(\ell) \qquad (\star\star\star)$$

- For case (iii), $\lambda < 0$, and hence, we write $\lambda = -\alpha^2$ so that the soltuion of equation $(\star\star\star)$ gives

$$X(x) = A\cos\alpha x + B\sin\alpha x \qquad (57)$$

whence, using the boundary condition, we derive the nontrivial soltuion

$$X(x) = B\sin\alpha x \qquad (58)$$

where B is an arbitrary nonzero constant. Clearly, since $B \neq 0$ and $X(\ell) = 0$,

$$\sin\alpha\ell = 0 \qquad (59)$$

which gives the solution for the parameter α

$$\alpha = \alpha_n = \left(\frac{n\pi}{\ell}\right), \quad n = 1, 2, 3, \dots \quad (60)$$

Note that $n = 0$, $(\alpha = 0)$ leads to the trivial solution $u(x,t) = 0$, and hence, the case $n = 0$ has to be excluded.
We see that there exists an infinite set of discrete values of α for which the problem has a nontrivial solution. These values α_n are called the *eigenvalues*, and the corresponding solutions are

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{\ell}\right). \qquad (61)$$

Example continued

We next solve

$$\frac{d^2T}{dt^2} = \lambda c^2 T, \qquad (\star\star\star\star)$$

with $\lambda = -\alpha_n^2$ to find the solution for $T_n(t)$ as

$$T_n(t) = C_n \cos(\alpha_n ct) + D_n \sin(\alpha_n ct) \qquad (62)$$

where C_n and D_n are constants of integration. Combining

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{\ell}\right) \quad T_n(t) = C_n \cos(\alpha_n ct) + D_n \sin(\alpha_n ct) \qquad (63)$$

yields the solution from $u(x,t) = X(x)T(t)$ as

$$u_n(x,t) = \left[a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right)\right] \sin\left(\frac{n\pi x}{\ell}\right), \qquad (64)$$

where $a_n = C_n B_n$, $b_n = B_n D_n$ are new arbitrary constants and $n = 1, 2, 3, \ldots$
These solution $u_n(x,t)$, corresponding to eigenvalues $\alpha_n = (\frac{n\pi}{\ell})$, are called the *eigenfunctions*.

Exemple continued

$$u_n(x,t) = \left[a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right)\right] \sin\left(\frac{n\pi x}{\ell}\right), \qquad (65)$$

Finally, since the problem is linear, the most general solution is obtained in the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi ct}{\ell} + b_n \sin\frac{n\pi ct}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right), \qquad (66)$$

provided the series converges and it is twice continuously differentiable with respect to x and t.
The arbitrary coefficients a_n and b_n are determined from the initial conditions

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right), \quad u_t(x,0) = g(x) = \left(\frac{\pi c}{\ell}\right) \sum_{n=1}^{\infty} n b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

These are the Fourier series expansions of the functions $f(x)$ and $g(x)$. Thus a_n and b_n are given by

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad b_n = \frac{2}{n\pi c}\int_0^{\ell} g(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

Hence the problem is completely solved.

Graphical representation of the solution

We examine now the graphical representation of the solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{\ell} + b_n \sin \frac{n\pi ct}{\ell} \right) \sin \left(\frac{n\pi x}{\ell} \right), \tag{67}$$

in the context of the free vibration of a string of length ℓ. The eigenfunctions

$$u_n(x,t) = \left(a_n \cos \frac{n\pi ct}{\ell} + b_n \sin \frac{n\pi ct}{\ell} \right) \sin \left(\frac{n\pi x}{\ell} \right), \qquad (✿)$$

are called the nth harmonic. The first harmonic ($n = 1$) is called the fundamental harmonic and all other harmonics ($n > 1$) are called *overtones*.
In order to describe waves produced in the plucked string with zero initial velocity ($u_t(x,0) = 0$), we write the solution (✿) in the form

$$u_n(x,t) = a_n \sin \left(\frac{n\pi x}{\ell} \right) \cos \left(\frac{n\pi ct}{\ell} \right), \quad n = 1, 2, 3 \ldots \tag{68}$$

Graphical representation of the solution

$$u_n(x,t) = a_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right), \quad n = 1, 2, 3\ldots \tag{69}$$

These solutions are called *standing waves* with amplitude $a_n \sin(\frac{n\pi x}{\ell})$ which vanishes at

$$x = 0, \frac{\ell}{n}, \frac{2\ell}{n}, \ldots, \ell.$$

These are called the *nodes* of the nth harmonic.

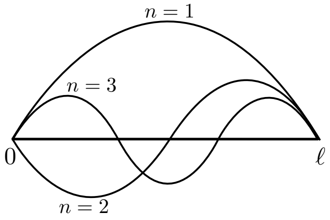


Fig. 1 Several harmonics of vibration in a string.

Outline

- Finite Difference Method
- Galerkin Method
- Finite Element Method

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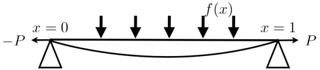
Unidimensional problem

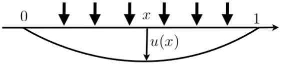
Assume that c and f are two continuous functions on the interval $[0,1]$. We want to find a function u twice continuously differentiable on $[0,1]$ such, that

$$-u''(x)+c(x)u(x)=f(x) \quad \text{for } 0<x<1, \quad u(0)=u(1)=0 \qquad (\star)$$

An example of physical situation for this problem is deflection of a beam of a unit length, of the ends $x=0$ and $x=1$, stretched along the axis by the force P, subjected to a linear charge density $f(x)$ and simply resting on its ends. Thus, the deflection moment $u(x)$ at the point x is a solution of the problem $(\star)$ with $c(x)=P/EI(x)$, where E is the Young's modulus of the material and $I(x)$ is the principal moment of inertia of a beam's section at the point x.



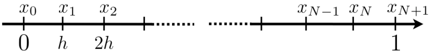
Another example is the vertical displacement $u(x)$ at the point x of a tightrope stretched between the ends $x=0$ and $x=1$ subjected to a unit tension and a vertical charge density $f(x)$; in this case, we have $c(x)=0$, $\forall x\in[0,1]$.



Finite Difference Method

$$-u''(x) + c(x)u(x) = f(x) \quad \text{for } 0 < x < 1, \quad u(0) = u(1) = 0. \qquad (\star)$$

Let N be an integer, $h = \dfrac{1}{(N+1)}$ and we note the discretization points as $x_j = jh$, $j = 0, 1, 2, \ldots, N+1$.



In order to find the numerical solution of the problem $(\star)$ by the finite difference method, we have to find the values u_j being the approximation of $u(x_j)$ and satisfying the following formula:

$$\begin{aligned} &\frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2} + c(x_j)u_j = f(x_j), \quad 1 \leqslant j \leqslant N, \\ &u_0 = u_{N+1} = 0 \end{aligned} \qquad (70)$$

The problem (70) is called the *discrete problem* of $(\star)$.

Finite Difference Method

If we suppose that u and f are two column vectors of size N, and A is the $N \times N$ three-diagonal matrix such, that:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}, f = \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ f_n \end{bmatrix}, A = \frac{1}{h^2} \begin{bmatrix} 2 + c_1 h^2 & -1 & 0 & \dots & 0 \\ -1 & 2 + c_2 h^2 & -1 & \dots & 0 \\ 0 & -1 & 2 + c_3 h^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \dots & -1 & 2 + c_N h^2 \end{bmatrix}$$

where $c_i = c(x_i)$, then the discrete problem is equivalent to the following linear system of N equations with N unknowns:

$$Au = f. \tag{71}$$

Theorem 1. Convergence. Assume that $c(x) \geqslant 0$, $\forall x \in [0, 1]$ and the solution u of $(\star)$ is four times continuously differentiable. Then existes a constant C independent of N such, that

$$\max_{1 \leqslant j \leqslant N} |u(x_j) - u_j| \leqslant Ch^2. \tag{72}$$

The weak formulation

Let us consider the problem $(\star)$ $-u''(x) + c(x)u(x) = f(x)$ for $0 < x < 1$, $u(0) = u(1) = 0$. We multiply the first equation by a function v continuously differentiable on the interval $[0, 1]$ and integrate on the interval $[0, 1]$:

$$-\int_0^1 u''(x)v(x)dx + \int_0^1 c(x)u(x)v(x)dx = \int_0^1 f(x)v(x)dx. \tag{73}$$

By integration by parts of the first term, we get

$$\int_0^1 u'(x)v'(x)dx - u'(1)v(1) + u'(0)v(0) + \int_0^1 c(x)u(x)v(x)dx = \int_0^1 f(x)v(x)dx. \tag{74}$$

If we suppose that the function v is null for $x = 0$ and $x = 1$, we get the following equality

$$\int_0^1 u'(x)v'(x)dx + \int_0^1 c(x)u(x)v(x)dx = \int_0^1 f(x)v(x)dx. \tag{$\star\star$}$$

Assume now, that V is a space of all the continuous functions g such, that its derivative g' is piecewise continuous and $g(0) = g(1) = 0$. Since the sum of two functions from V is also the function of V, as well as, the product of a function from V by a real number, thus V is a vector space.

The weak formulation

Now the problem

$$-u''(x) + c(x)u(x) = f(x) \quad \text{for } 0 < x < 1, \quad u(0) = u(1) = 0, \qquad (\star)$$

can be written in the equivalent form: Find $u \in V$ such that

$$\int_0^1 u'(x)v'(x)dx + \int_0^1 c(x)u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \ \forall v \in V. \qquad (\star\star)$$

- The problem $(\star\star)$ is called *weak problem* or *variational problem*.
- The solutions u of the weak problem $(\star\star)$ are less regular as the solutions to the differential problem $(\star)$, thus the solution to the weak problem is called the *weak solution*.
- However, the solution of the differential problem $(\star)$ is the solution of the weak problem $(\star\star)$.

The Galerkin method

We present now the Galerkin method which is based on the weak formulation

$$\begin{cases} \text{Find } u \in V \text{ such, that:} \\ \displaystyle\int_0^1 u'(x)v'(x)dx + \int_0^1 c(x)u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \ \forall v \in V. \end{cases} \tag{$\star\star$}$$

and is the departure point for the finite element method.

Let us suppose that $\varphi_1, \varphi_2, \ldots, \varphi_N$ are N functions linearly independent in the vector space V, then we can construct a subspace V_h spanned on the linear combination of the functions $\varphi_1, \varphi_2, \ldots, \varphi_N$. Thus, V_h is a space of the functions g which can be expressed as a linear combination of the basic functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ as follows:

$$g(x) = \sum_{i=1}^{N} g_i \varphi_i(x), \quad \text{where } g_i \text{ are } N \text{ real numbers.} \tag{75}$$

So, if we suppose that u_h and v_h are also the function from the subspace V_h, then we can formulate an approximation of the problem $(\star\star)$ as follows:

$$\begin{cases} \text{Find } u_h \in V_h \text{ such, that:} \\ \displaystyle\int_0^1 u_h'(x)v_h'(x)dx + \int_0^1 c(x)u_h(x)v_h(x)dx = \int_0^1 f(x)v_h(x)dx, \end{cases} \tag{$\star\star\star$}$$

The Galerkin method

$$\left\{ \begin{array}{l} \text{Find } u_h \in V_h \text{ such, that for any } v_h \in V_h : \\ \displaystyle\int_0^1 u_h'(x) v_h'(x) dx + \int_0^1 c(x) u_h(x) v_h(x) dx = \int_0^1 f(x) v_h(x) dx, \end{array} \right. \tag{$\star\star\star$}$$

Since the function $u_h \in V_h$, then it can be written as a linear combination of the functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ as:

$$u_h(x) = \sum_{i=1}^{N} u_i \varphi_i(x), \tag{76}$$

where $u_1, u_2, \ldots, u_N$ are N real numbers to be determined. Taking $v_h = \varphi_j$, $1 \leqslant j \leqslant N$ in $(\star\star\star)$, then the problem $(\star\star\star)$ is equivalent to the following problem:

$$\left\{ \begin{array}{l} \text{Find the coefficients } u_1, u_2, \ldots, u_N \text{ such, that:} \\ \displaystyle\sum_{i=1}^{N} u_i \left(\int_0^1 \varphi_i'(x) \varphi_j'(x) dx + \int_0^1 c(x) \varphi_i(x) \varphi_j(x) dx \right) = \int_0^1 f(x) \varphi_j(x) dx, \\ \text{for all } j = 1, 2, \ldots, N. \end{array} \right. \tag{77}$$

The Galerkin method

$$\begin{cases} \text{Find the coefficients } u_1, u_2, \ldots, u_N \text{ such, that:} \\ \displaystyle\sum_{i=1}^{N} u_i \left(\int_0^1 \varphi_i'(x)\varphi_j'(x)dx + \int_0^1 c(x)\varphi_i(x)\varphi_j(x)dx \right) = \int_0^1 f(x)\varphi_j(x)dx, \\ \text{for all } j = 1, 2, \ldots, N. \end{cases} \tag{78}$$

We denote by $A = A_{ji}$ a $N \times N$ matrix of coefficients:

$$A_{ji} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx + \int_0^1 c(x)\varphi_i(x)\varphi_j(x)dx, \tag{79}$$

by u we note a vector of N components $u_1, u_2, \ldots, u_N$ and by $f = f_j$, a vector of N components:

$$f_j = \int_0^1 f(x)\varphi_j(x)dx. \tag{80}$$

Thus, the problem (77) can be rewritten in a form of a linear system:

$$Au = f.$$

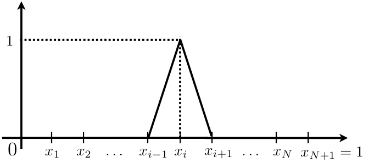
Finite Element Method of the first order

$$-u''(x) + c(x)u(x) = f(x) \quad \text{for } 0 < x < 1, \quad u(0) = u(1) = 0. \qquad (\star)$$

Let us divide the interval $[0,1]$ by $N+1$ parts (N is an integer) and take $h = \dfrac{1}{N+1}$, $x_i = ih$ with $i = 0, 1, 2, \ldots, N+1$.

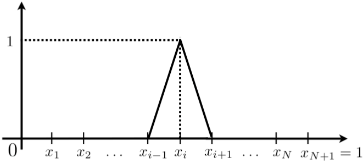
We define, for $i = 1, 2, \ldots, N$ the following functions:

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \leqslant x \leqslant x_i, \\[2ex] \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & \text{if } x_i \leqslant x \leqslant x_{i+1}, \\[2ex] 0 & \text{if } x \leqslant x_{i-1} \text{ or } x \geqslant x_{i+1}. \end{cases}$$



Finite Element Method of the first order

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \leqslant x \leqslant x_i, \\ \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & \text{if } x_i \leqslant x \leqslant x_{i+1}, \\ 0 & \text{if } x \leqslant x_{i-1} \text{ or } x \geqslant x_{i+1}. \end{cases}$$



Functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ are linearly independent, so we choose them to span the subspace V_h of the vector space V.

Therefore, we say that:

- $x_0, x_1, x_2, \ldots, x_{N+1}$ are the *nodes of the discretization*,
- $[x_0, x_1], [x_1, x_2], [x_N, x_{N+1}]$ are the *geometrical elements*,
- $\varphi_1, \varphi_2, \ldots, \varphi_N$ are the base functions of subspace V_h of type *finite elements of first order* associated with the interior nodes $x_1, x_2, \ldots, x_N$.

The finite element method is the Galerkin method with the functions φ_i, $i = 1, 2, \ldots, N$ defined above.

Finite Element Method of the first order

The boundary value problem:

$$-u''(x) + c(x)u(x) = f(x) \quad \text{for } 0 < x < 1, \quad u(0) = u(1) = 0. \qquad (\star)$$

The weak formulation of the problem $(\star)$:

$$\begin{cases} \text{Find } u \in V \text{ such, that:} \\ \displaystyle\int_0^1 u'(x)v'(x)dx + \int_0^1 c(x)u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \ \forall v \in V. \end{cases} \qquad (\star\star)$$

The corresponding finite element method is

$$\begin{cases} \text{Find } u_h \in V_h \text{ such, that for any } v_h \in V_h : \\ \displaystyle\int_0^1 u_h'(x)v_h'(x)dx + \int_0^1 c(x)u_h(x)v_h(x)dx = \int_0^1 f(x)v_h(x)dx. \end{cases} \qquad (\star\star\star)$$

Finite Element Method of the first order

Writing $u_h(x) = \sum_{i=1}^{N} u_i \varphi_i(x)$, and taking $v_h = \varphi_j$, $j = 1, 2, \ldots, N$, we see that the finite element method is equivalent to the following linear system for the unknowns $u_1, u_2, \ldots, u_N$:

$$\begin{cases} \text{Find the coefficients } u_1, u_2, \ldots, u_N \text{ such, that:} \\ \sum_{i=1}^{N} u_i \left(\int_0^1 \varphi_i'(x)\varphi_j'(x)dx + \int_0^1 c(x)\varphi_i(x)\varphi_j(x)dx \right) = \int_0^1 f(x)\varphi_j(x)dx, \\ \text{for all } j = 1, 2, \ldots, N. \end{cases} \tag{81}$$

Finite Element Method of the first order

$$\left\{\begin{array}{l}\text{Find the coefficients } u_1, u_2, \dots, u_N \text{ such, that:}\\ \displaystyle\sum_{i=1}^{N} u_i\left(\int_0^1 \varphi_i'(x)\varphi_j'(x)dx + \int_0^1 c(x)\varphi_i(x)\varphi_j(x)dx\right) = \int_0^1 f(x)\varphi_j(x)dx,\\ \text{for all } j = 1, 2, \dots, N.\end{array}\right. \tag{82}$$

And if we denote by $A = A_{ji}$ a $N \times N$ matrix of coefficients:

$$A_{ji} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx + \int_0^1 c(x)\varphi_i(x)\varphi_j(x)dx, \tag{83}$$

by u a vector of N components $u_1, u_2, \dots, u_N$ and by $f = f_j$, a vector of N components:

$$f_j = \int_0^1 f(x)\varphi_j(x)dx, \tag{84}$$

thus, the problem (82) can be rewritten in a form of the linear system:

$$Au = f.$$

Finite Element Method of the first order

Calculation of the coefficients $A = A_{ji}$, and $f = f_j$:

We have: $A_{ji} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx + \int_0^1 c(x)\varphi_i(x)\varphi_j(x)dx.$

It is easy to verify that

$$\int_0^1 \varphi_i'(x)\varphi_j'(x)dx = \begin{cases} 2/h & \text{if } i = j, \\ -1/h & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{85}$$

In order to find the values of $\int_0^1 c(x)\varphi_i(x)\varphi_j(x)dx$ and $\int_0^1 f(x)\varphi_j(x)dx$, we can use the following formula of numerical integration:

$$\int_0^1 \ell(x)dx = h\left(\frac{1}{2}\ell(x_0) + \ell(x_1) + \ell(x_1) + \cdots + \ell(x_N) + \frac{1}{2}\ell(x_{N+1})\right). \tag{86}$$

Thus, we get

$$\int_0^1 c(x)\varphi_i(x)\varphi_j(x)dx = \begin{cases} hc(x_j) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \int_0^1 f(x)\varphi_j(x)dx = hf(x_j).$$