Computer Modeling

Ordinary differential equations
(ODE)

A solution of ODE

An ordinary differential equation of the *n*–th order

$G\left(t, y(t), y'(t), \cdots, y^{(n)}(t)\right) = 0$

A solution $y(t) = \varphi(t)$ in an interval t $\in (b, d) \subset \mathbb{R}$:

$\varphi(t)$ is *n*-times differentiable on (b, d) and satisfies
$G\left(t, \varphi(t), \varphi'(t), \cdots, \varphi^{(n)}(t)\right) = 0.$

$\varphi(t)$ is also called a trajectory.

(we allow $b = -\infty$ and $d = \infty$)

Uniqueness of a solution

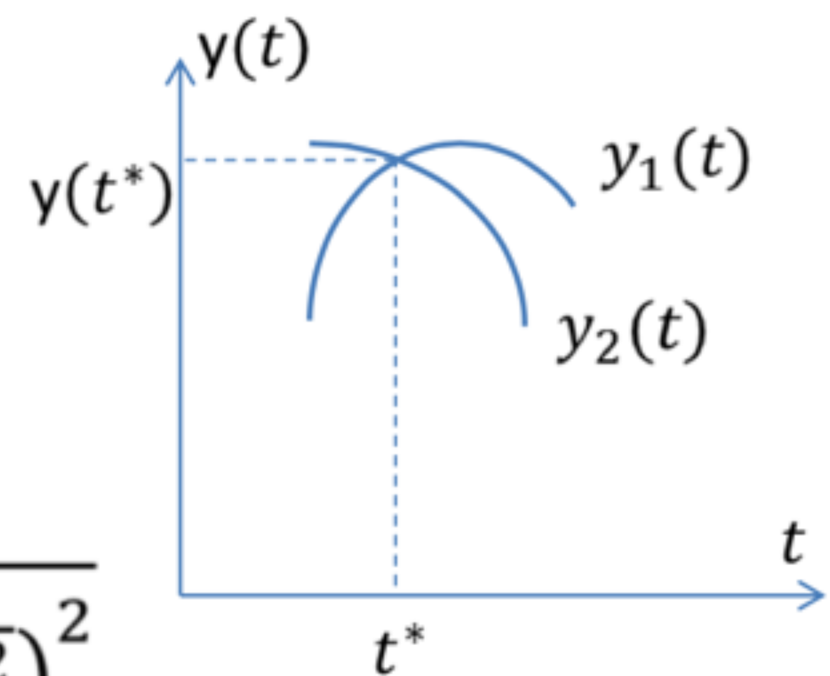
A solution is unique, if for any point $(t^*, y^*(t^*))$ there is only one curve y(t) going through this point.

A solution may be non-unique,

e.g. for the following equation

$$y(t)\sqrt{1-t^2}y'(t) = t\sqrt{1-y^2(t)}$$

two solutions go through the point (0,1):



$$y_1(t) = 1 \text{ and } y_2(t) = \sqrt{1-\left(1-\sqrt{1-t^2}\right)^2}$$

$$y_1'(t) = 0 \text{ and } y_2'(t) = \frac{t\left(1-\sqrt{1-t^2}\right)}{\sqrt{1-t^2}\sqrt{1-\left(1-\sqrt{1-t^2}\right)^2}} = \frac{t\left(1-\sqrt{1-t^2}\right)}{\sqrt{1-t^2}\,y_2(t)}$$

Types of solutions

A solution may depend on an arbitrary constant, e.g. an equation $y'(t) = h(t)$ has a solution $y(t) = \int h(t)dt + C$.

Consider a domain D containing only points $(t, y) \in D$ such that the solution $y(t)$ going through any point of D is unique. A function $y(t) = \varphi(t, C)$ is called a general solution of the 1st order equation, if:

(i) φ is continuously differentiable with respect to t,

(ii) it is possible to find C for any points $(t, y) \in D$,

(iii) $\varphi(t, C)$ is a solution for any such determined constant.

A general solution of an *n*-th order equation depends on *n* arbitrary constants $C_1, \cdots, C_n$.

A particular solution goes through a particular point (t, y) and can be obtained by setting particular values to the constants.

A singular solution is nonunique in all points (t, y) of its trajectory.

An example

$$y'(t) = 2\sqrt{y(t)}, \qquad y \geq 0$$

$y(t) = 0$ is a solution

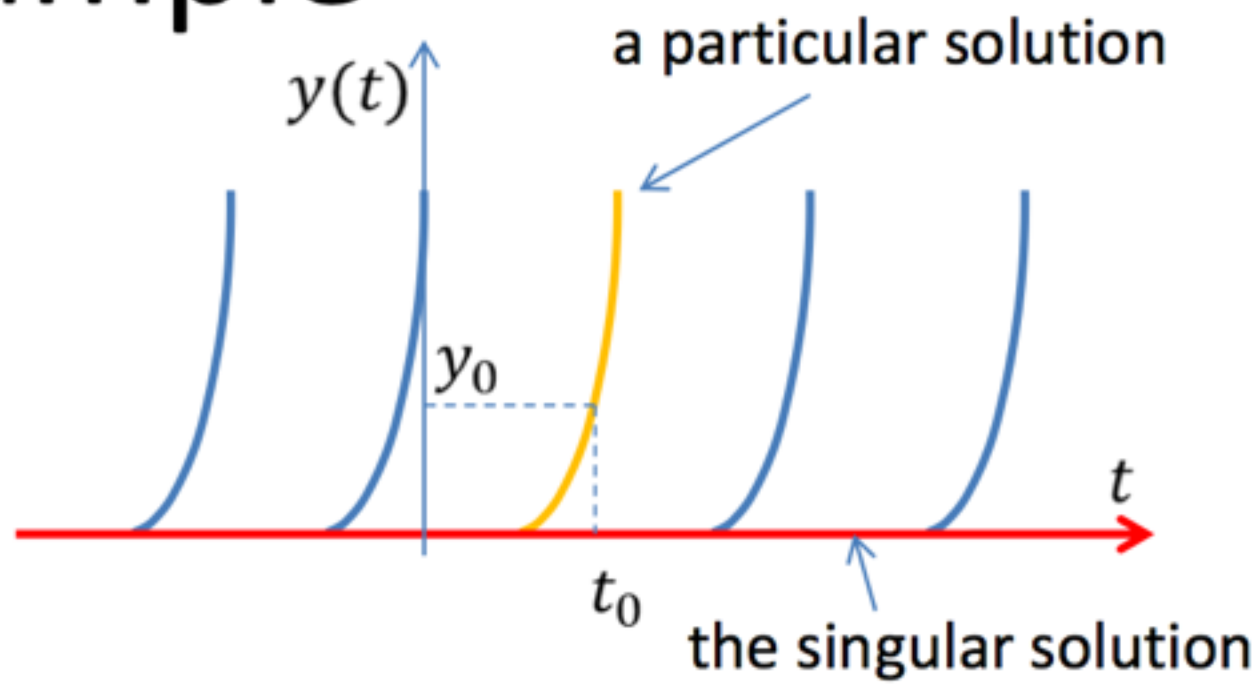
assume $y(t) > 0$ for any t

$$\frac{y'}{2\sqrt{y}} = 1 \ \rightarrow (\sqrt{y})' = 1$$

$$\sqrt{y} = t + C, \qquad C > -t$$

all solutions

$$y(t) = (t + C)^2, \qquad C \geq -t$$



$D = \{(t, y): -\infty < t < \infty, y > 0\}$

A general solution:
$y(t) = (t + C)^2, \qquad t > -C$

A particular solution going through (t_0, y_0):
$y(t) = (t - t_0 + \sqrt{y_0})^2$

The singular solution $\quad y(t) = 0$
(nonunique at all points of its trajectory)

Initial and boundary conditions

Initial value problem:
conditions for operators $A_i(t, y) = 0,\ i = 1, \cdots, n$ at a t_0 such that $(t_0, y(t_0)) \in D$

Cauchy problem:

$$y(t_0) = y_0^{(0)}, y'(t_0) = y_1^{(0)}, \cdots,$$

$$y^{(n-1)}(t_0) = y_{n-1}^{(0)}, \qquad (t_0, y(t_0)) \in D$$

Boundary value problem: conditions distributed among two different points t_o and t_1

System of ODE equations

$$G\left(t, y(t), y'(t), \cdots, y^{(m)}(t)\right) = 0$$

Denote

$y_1(t) = y(t)$
$y_2(t) = y'(t)$
$\cdots\cdots\cdots\cdots\cdots$
$y_m(t) = y^{(m-1)}(t)$

then

$y_1'(t) = y_2(t)$
$y_2'(t) = y_3(t)$
$\cdots\cdots\cdots\cdots\cdots$
$y_{m-1}'(t) = y_m(t)$
$G(t, y_1(t), y_2(t), \cdots, y_m'(t))=0$

If the equation

$G(t, y_1(t), y_2(t), \cdots, y_m'(t))=0$

can be solved for $y_m'(t)$, then the equation can be written in the normal form. In general

$y_1'(t) = f_1(t, y_1, y_2, \cdots, y_M)$
$y_2'(t) = f_2(t, y_1, y_2, \cdots, y_M)$
$\cdots\cdots\cdots\cdots\cdots$
$y_{M-1}'(t) = f_{M-1}(t, y_1, y_2, \cdots, y_M)$
$y_M'(t) = f_M(t, y_1, y_2, \cdots, y_M)$

System of linear equations

System of linear equations:

$$y_1'(t) = a_{11}(t)y_1(t) + \cdots + a_{1M}(t)y_M(t) + u_1(t)$$
$$y_2'(t) = a_{21}(t)y_1(t) + \cdots + a_{2M}(t)y_M(t) + u_2(t)$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$
$$y_M'(t) = a_{1M}(t)y_1(t) + \cdots + a_{MM}(t)y_M(t) + u_M(t)$$

Picard theorem. If $a_{ij}(t), i, j = 1,2,\cdots, M$ and $u_i(t), i = 1,2,\cdots, M$ are continuous in an interval $[b, d]$, then the above system has the unique solution $y_1(t), y_2(t), \cdots, y_M(t)$, which satisfies initial conditions $y_1^{(0)}, y_2^{(0)}, \cdots, y_M^{(0)}$ at any $t \in [b, d]$ for arbitrary values $y_1^{(0)}, y_2^{(0)}, \cdots, y_M^{(0)}$. The solution exists and is continuously differentiable in (b, d). → no singular solutions

Homogeneous and nonhomogeneous systems

The system of equations is called homogeneous, if

$$u_1(t) = u_2(t) = \cdots = u_M(t) \equiv 0$$

Otherwise it is called nonhomogeneous.

Homogeneous system

$$y_1'(t) = a_{11}(t)y_1(t) + \cdots + a_{1M}(t)y_M(t)$$
$$y_2'(t) = a_{21}(t)y_1(t) + \cdots + a_{2M}(t)y_M(t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (*)$$
$$y_M'(t) = a_{1M}(t)y_1(t) + \cdots + a_{MM}(t)y_M(t)$$

Assumption: $a_{ij}(t)$ are continuous in $[b, d]$

If any column of functions ($1 \le m \le M$)

$y_{11}(t), \cdots\cdots\cdots, y_{m1}(t)$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$y_{1M}(t), \cdots\cdots\cdots, y_{mM}(t)$

is a solution of (*), then their linear combination

$$y_1(t) = h_1 y_{11}(t) + \cdots\cdots\cdots + h_m y_{m1}(t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_M(t) = h_1 y_{1M}(t) + \cdots\cdots\cdots + h_m y_{mM}(t)$$

is also a solution (proof by insertion to (*))

Sketch of a proof ($M = 2$)

$$y_1'(t) = a_{11}(t)y_1(t) + a_{12}(t)y_2(t)$$
$$y_2'(t) = a_{21}(t)y_1(t) + a_{22}(t)y_2(t)$$

Solutions

$y_{11}(t),\ y_{21}(t)$

$y_{12}(t),\ y_{22}(t)$ i.e.

$$y_{11}'(t) = a_{11}(t)y_{11}(t) + a_{12}(t)y_{12}(t)$$
$$y_{12}'(t) = a_{21}(t)y_{11}(t) + a_{22}(t)y_{12}(t)$$
$$y_{21}'(t) = a_{11}(t)y_{21}(t) + a_{12}(t)y_{22}(t)$$
$$y_{12}'(t) = a_{21}(t)y_{21}(t) + a_{22}(t)y_{22}(t)$$

Linear combinations

$$\tilde{y}_1(t) = h_1 y_{11}(t) + h_2 y_{21}(t)$$
$$\tilde{y}_2(t) = h_1 y_{12}(t) + h_2 y_{22}(t)$$

Differentiating the first equation

$$\tilde{y}_1'(t) = h_1 y_{11}'(t) + h_2 y_{21}'(t) =$$
$$= h_1 [a_{11}(t)y_{11}(t) + a_{12}(t)y_{12}(t)] +$$
$$+ h_2 [a_{11}(t)y_{21}(t) + a_{12}(t)y_{22}(t)] =$$
$$= a_{11}(t)[h_1 y_{11}(t) + h_2 y_{21}(t)] +$$
$$+ a_{12}(t)[h_1 y_{12}(t) + h_2 y_{22}(t)] =$$
$$= a_{11}(t)\tilde{y}_1 + a_{12}(t)\tilde{y}_2$$

Similarly for the second equation.

Linear dependence

M sets of functions

$y_{11}(t), \cdots\cdots\cdots, y_{M1}(t)$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (**)

$y_{1M}(t), \cdots\cdots\cdots, y_{MM}(t)$

are called linearly dependent in an interval (b, d), if and only if there exists a set of numbers $h_1, \cdots, h_M$, not all zero, such that for all $t \in (b, d)$

$h_1 y_{11}(t) + \cdots\cdots\cdots + h_M y_{1M}(t) = 0$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$h_1 y_{M1}(t) + \cdots\cdots\cdots + h_M y_{MM}(t) = 0$

If such numbers do not exist, then the sets of functions are linearly independent.

Example (£)

M=3 sets of functions, with
$\lambda_i \neq \lambda_j$ for $i \neq j$,
$s_{11}e^{\lambda_1 t}, s_{21}e^{\lambda_1 t}, s_{31}e^{\lambda_1 t}$
$s_{12}e^{\lambda_2 t}, s_{22}e^{\lambda_2 t}, s_{32}e^{\lambda_2 t}$
$s_{13}e^{\lambda_3 t}, s_{23}e^{\lambda_3 t}, s_{33}e^{\lambda_3 t}$
and with at least one nonzero s_{ij} in each row, are linearly independent in the interval $(-\infty, \infty)$.

Consider

$$h_1 s_{11}e^{\lambda_1 t} + h_2\, s_{12}e^{\lambda_2 t} + h_3 s_{13}e^{\lambda_3 t} = 0$$
$$h_1 s_{21}e^{\lambda_1 t} + h_2\, s_{22}e^{\lambda_2 t} + h_3 s_{23}e^{\lambda_3 t} = 0$$
$$h_1 s_{31}e^{\lambda_1 t} + h_2 s_{32}e^{\lambda_2 t} + h_3 s_{33}e^{\lambda_3 t} = 0$$

<u>Sketch of a proof</u>

Assume $s_{13} \neq 0$. Consider the first equation

$$h_1 s_{11}e^{\lambda_1 t} + h_2 s_{12}e^{\lambda_2 t} + h_3 s_{13}e^{\lambda_3 t} = 0$$

Multiplying both sides by $e^{-\lambda_1 t}$

$$h_1 s_{11} + h_2 s_{12}e^{(\lambda_2-\lambda_1)t} + h_3 s_{13}e^{(\lambda_3-\lambda_1)t} = 0$$

Differentiating

$$h_2 s_{12}(\lambda_2 - \lambda_1)e^{(\lambda_2-\lambda_1)t} + h_3 s_{13}(\lambda_3 - \lambda_1)e^{(\lambda_3-\lambda_1)t} = 0$$

Multiplying by $e^{-(\lambda_2-\lambda_1)t}$ and differentiating

$$h_3 s_{13}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)e^{(\lambda_3-\lambda_2)t} = 0$$

As $s_{13} \neq 0$, $\lambda_3 \neq \lambda_1$, $\lambda_3 \neq \lambda_2$, then $h_3 = 0$.

Similarly for other equations.

Valid for any integer $M > 0$.

The Wronskian

Theorem

If the sets of functions (**) are linearly independent, i.e. there do not exist nonzero values of $h_1, \cdots, h_M$ satisfying the equations

$$h_1 y_{11}(t) + \cdots\cdots\cdots + h_M y_{1M}(t) = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (\#)$$

$$h_1 y_{M1}(t) + \cdots\cdots\cdots + h_M y_{MM}(t) = 0$$

and each set is a solution of the homogeneous system of equation (*), then the determinant

$$W(t) = \begin{vmatrix} y_{11}(t) & \cdots & y_{1M}(t) \\ \vdots & \ddots & \vdots \\ y_{M1}(t) & \cdots & y_{MM}(t) \end{vmatrix} \neq 0$$

for any $t \in (b, d)$.

$W(t)$ – the Wronskian (determinant)

Proof

Assume that (**) are linearly independent and there exist $t_0 \in (b, d)$ that $W(t_0) = 0$. Then there exist not all zero values of $h_1^0, \cdots, h_M^0$ satisfying (#) at $t = t_0$. Consider the functions

$y_m(t) = \sum_{i=1}^{M} h_i^0 y_{im}(t), \;\; m = 1, \ldots, M.$

They are solutions of (*) and $y_m(t_0) = 0$. But $y_m(t) \equiv 0$ are also solutions. From the Picard theorem the solutions are unique (zero). Thus (#) are satisfied with $h_i = h_i^0, i = 1, \ldots, M$, not all equal zero. Finally, the sets of functions (**) are dependent, what contradicts the assumption.

Conclusions

If $W(t_0) = 0$ at any $t_0 \in (b, d)$, then $W(t) \equiv 0$ for any $t \in (b, d)$.

If $W(t) \neq 0$ at any $t \in (b, d)$, then it is different form zero in all points of (b, d).

General solutions

M linearly independent solutions of a homogeneous system of equations

$$y_{11}(t), \cdots\cdots\cdots, y_{M1}(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (**)$$

$$y_{1M}(t), \cdots\cdots\cdots, y_{MM}(t)$$

are called a fundamental system of solutions.

An example

$$\frac{dy_1}{dt} = \frac{t^2}{1+t^2} y_1 + \frac{1}{1+t^2} y_2$$

$$\frac{dy_2}{dt} = \frac{t^2 - 2t}{1+t^2} y_1 + \frac{2t+1}{1+t^2} y_2$$

$y_{11} = e^t$, $y_{21} = -1$

$y_{12} = e^t$, $y_{22} = t^2$

2 sets of solutions

<u>Fundamental theorem</u>

If (**) is a fundamental system of solutions of a homogeneous system of equations in (b, d), then

$$y_1(t) = h_1 y_{11}(t) + \cdots\cdots\cdots + h_M y_{1M}(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_M(t) = h_1 y_{M1}(t) + \cdots\cdots\cdots + h_M y_{MM}(t)$$

is a set of general solutions in the domain $b < t < d, -\infty < y_m < \infty$, $m = 1, \ldots, M$, for an arbitrary set of numbers $h_1, \cdots, h_M$.

$$W(t) = \begin{vmatrix} e^t & -1 \\ e^t & t^2 \end{vmatrix} = e^t(t^2+1) > 0$$

$y_1(t) = h_1 e^t - h_2$

$y_2(t) = h_1 e^t + h_2 t^2$

a general solution

Matrix equation

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \cdots \\ a_{M1} & \cdots & a_{MM} \end{bmatrix}$$

$$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}(t) + \boldsymbol{u}(t)$$

$$\frac{d\boldsymbol{y}(t)}{dt} = \begin{bmatrix} \frac{dy_1(t)}{dt} \\ \vdots \\ \frac{dy_M(t)}{dt} \end{bmatrix}$$

$$y_1'(t) = a_{11}(t)y_1(t) + a_{12}(t)y_2(t)$$
$$y_2'(t) = a_{21}(t)y_1(t) + a_{22}(t)y_2(t)$$
$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

a homogeneous matrix equation

$$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}(t)$$

a general solution

$$\boldsymbol{y}(t) = \mathbf{Y}(t)\boldsymbol{h}$$

$$\mathbf{Y}(t) = \begin{bmatrix} y_{11}(t) & \cdots & y_{1M}(t) \\ \vdots & \ddots & \cdots \\ y_{M1}(t) & \cdots & y_{MM}(t) \end{bmatrix}$$

a fundamental (solution) matrix [nonsingular, Wronskian positive]

$$\boldsymbol{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_M \end{bmatrix}$$

Derivative of matrices

$$\frac{d\mathbf{Y}(t)}{dt} = \begin{bmatrix} \frac{dy_{11}(t)}{dt} & \cdots & \frac{dy_{1M}(t)}{dt} \\ \vdots & \ddots & \cdots \\ \frac{dy_{M1}(t)}{dt} & \cdots & \frac{dy_{MM}(t)}{dt} \end{bmatrix}$$

A – a constant matrix

$$\frac{d\mathbf{A}}{dt} = \mathbf{0}$$

$$\frac{d(\mathbf{A}\mathbf{Y}(t))}{dt} = \mathbf{A}\frac{d\mathbf{Y}(t)}{dt}$$

$$\frac{d(\boldsymbol{Y}_1(t)+\boldsymbol{Y}_2(t))}{dt} = \frac{d\boldsymbol{Y}_1(t)}{dt} + \frac{d\boldsymbol{Y}_2(t)}{dt}$$

$$\frac{d(\boldsymbol{Y}_1(t)\boldsymbol{Y}_2(t))}{dt} = \\ = \frac{d\boldsymbol{Y}_1(t)}{dt}\boldsymbol{Y}_2(t) + \boldsymbol{Y}_1(t)\frac{d\boldsymbol{Y}_2(t)}{dt}$$

$\boldsymbol{X}_1\boldsymbol{X}_2 \neq \boldsymbol{X}_2\boldsymbol{X}_1$ matrix multiplication is not commutative

$$\frac{d\boldsymbol{Y}^2(t)}{dt} = \frac{d\boldsymbol{Y}(t)}{dt}\boldsymbol{Y}(t) + \boldsymbol{Y}(t)\frac{d\boldsymbol{Y}(t)}{dt}$$

Thus

$\frac{d\boldsymbol{Y}^2(t)}{dt} = 2\boldsymbol{Y}(t)\frac{d\boldsymbol{Y}(t)}{dt}$ only if $\boldsymbol{Y}(t)$ and $\frac{d\boldsymbol{Y}(t)}{dt}$ commute

$$\boldsymbol{Y}(t)\boldsymbol{Y}^{-1}(t)=\mathbf{I}$$

$$\frac{d\boldsymbol{Y}(t)}{dt}\boldsymbol{Y}^{-1}(t) + \boldsymbol{Y}(t)\frac{d\boldsymbol{Y}^{-1}(t)}{dt} = \mathbf{0}$$

$$\frac{d\boldsymbol{Y}^{-1}(t)}{dt} = \boldsymbol{Y}^{-1}(t)\frac{d\boldsymbol{Y}(t)}{dt}\boldsymbol{Y}^{-1}(t)$$

Normed fundamental matrix

M equations for M solutions

$$\frac{d\mathbf{Y}(t)}{dt} = \mathbf{A}(t)\mathbf{Y}(t) \qquad (\&)$$

$$\frac{d\mathbf{Y}(t)}{dt} = \begin{bmatrix} \frac{dy_{11}(t)}{dt} & \cdots & \frac{dy_{1M}(t)}{dt} \\ \vdots & \ddots & \cdots \\ \frac{dy_{M1}(t)}{dt} & \cdots & \frac{dy_{MM}(t)}{dt} \end{bmatrix}$$

normed fundamental matrix
(state transition matrix)

$$\mathbf{\Phi}(t, t_0) = \mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)$$

$\boxed{\mathbf{\Phi}(t_0, t_0) = \mathbf{I}}$ – identity matrix

$\mathbf{\Phi}(t, t_0)$ is a solution of (&) with the initial condition $\mathbf{Y}(t_0) = \mathbf{I}$

$\mathbf{Z}(t) = \mathbf{\Phi}(t, t_0)\mathbf{H}$ $\quad$ $\mathbf{H}$ – nonsingular

$$\frac{d\mathbf{Z}(t)}{dt} = \frac{d(\mathbf{\Phi}(t, t_0)\mathbf{H})}{dt} = \frac{d\big(\mathbf{\Phi}(t, t_0)\big)}{dt}\mathbf{H} =$$
$$= \mathbf{A}(t)\,\mathbf{\Phi}(t, t_0)\mathbf{H} = \mathbf{A}(t)\mathbf{Z}(t)$$

$\mathbf{Z}(t)$ - a fundamental matrix

$\mathbf{Y}(t) = \mathbf{\Phi}(t, t_0)\mathbf{Y}(t_0)$ – a fundamental matrix with the initial condition $\mathbf{Y}(t_0)$

$$\mathbf{Y}(t) = \mathbf{\Phi}(t, t_1)\mathbf{Y}(t_1)$$
$$\mathbf{Y}(t) = \mathbf{\Phi}(t, t_1)\mathbf{\Phi}(t_1, t_0)\mathbf{Y}(t_0)$$

$$\boxed{\begin{aligned} \mathbf{\Phi}(t, t_0) &= \mathbf{\Phi}(t, t_1)\mathbf{\Phi}(t_1, t_0) \\ \mathbf{\Phi}^{-1}(t_1, t_0) &= \mathbf{\Phi}(t_0, t_1) \end{aligned}}$$

$t = t_0$ above

The solution satisfying the initial cond. $\boldsymbol{y}(t_0)$

$$\boldsymbol{y}(t) = \mathbf{\Phi}(t, t_0)\,\boldsymbol{y}(t_0)$$

Nonhomogeneous equations

$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}(t) + \boldsymbol{u}(t)$ ($)

Let's try a solution

$\boldsymbol{y}(t) = \boldsymbol{y}^s(t) + \boldsymbol{y}^0(t)$

$\boldsymbol{y}^s(t)$ – a particular solution of ($)

$\boldsymbol{y}^0(t) = \mathbf{Y}(t)\boldsymbol{h}$ – a general solution of the homogeneous equation

$$\frac{d\boldsymbol{y}^0(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}^0(t)$$

$$\frac{d\left(\boldsymbol{y}^s(t)+\boldsymbol{y}^0(t)\right)}{dt} = \mathbf{A}(t)\boldsymbol{y}^s(t) +$$

$$+\mathbf{A}(t)\boldsymbol{y}^0(t) + \boldsymbol{u}(t)$$

A general solution of
a nonhomogenous equation =
a particular solution of the
nonhomogenous equation + a general
solution of the homogenous equation

Variation of parameters method

$\boldsymbol{y}^s(t) = \mathbf{Y}(t)\boldsymbol{h}(t)$ $\mathbf{Y}(t)$ - a fundamental matrix

$$\frac{d\mathbf{Y}(t)}{dt}\boldsymbol{h}(t) + \mathbf{Y}(t)\frac{d\boldsymbol{h}(t)}{dt} = \mathbf{A}(t)\mathbf{Y}(t)\boldsymbol{h}(t) + \boldsymbol{u}(t)$$

$$\mathbf{Y}(t)\frac{d\boldsymbol{h}(t)}{dt} = \boldsymbol{u}(t)$$

$$\frac{d\boldsymbol{h}(t)}{dt} = \mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)$$

$$\boldsymbol{h}(t) = \int_{t_0}^{t} \mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau \qquad \boldsymbol{h}(t_0) = \mathbf{0}$$

$$\boldsymbol{y}^s(t) = \mathbf{Y}(t)\int_{t_0}^{t} \mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau$$

$$\boldsymbol{y}(t) = \mathbf{Y}(t)\boldsymbol{h} + \mathbf{Y}(t)\int_{t_0}^{t} \mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau$$

General solution of the nonhomogenous equation

General solution of nonhomogeneous equations

$$\boldsymbol{y}^s(t) = \mathbf{Y}(t)\int_{t_0}^{t}\mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau$$

$$\boldsymbol{y}(t) = \mathbf{Y}(t)\boldsymbol{h} + \mathbf{Y}(t)\int_{t_0}^{t}\mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau$$

$$\boldsymbol{y}(t) = \mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)\mathbf{Y}(t_0)\boldsymbol{h} +$$

$$+\mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)\int_{t_0}^{t}\left(\mathbf{Y}(\tau)\mathbf{Y}^{-1}(t_0)\right)^{-1}\boldsymbol{u}(\tau)d\tau =$$

$$= \boldsymbol{\Phi}(t,t_0)\boldsymbol{b} + \boldsymbol{\Phi}(t,t_0)\int_{t_0}^{t}\boldsymbol{\Phi}^{-1}(\tau,t_0)\boldsymbol{u}(\tau)d\tau$$

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t,t_0)\boldsymbol{b} + \int_{t_0}^{t}\boldsymbol{\Phi}(t,\tau)\boldsymbol{u}(\tau)d\tau$$

$\boldsymbol{\Phi}(t,\tau)$ or $\mathbf{Y}(t)$ has to be known!

$$\boldsymbol{b} = \mathbf{Y}(t_0)\boldsymbol{h}$$

$$\frac{dy(t)}{dt} = a(t)y(t) \qquad y(t_0) = y_0$$

$$\frac{dy}{y} = a(t) \qquad y(t) = y_0\, e^{\int_{t_0}^{t} a(\tau)d\tau}$$

A question

$$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}(t)$$

Is it true that $\boldsymbol{y}(t) = e^{\int_{t_0}^{t}\mathbf{A}(\tau)d\tau}\boldsymbol{y}_0$?

$$\int \mathbf{A}dt = \begin{bmatrix} \int a_{11}(t)dt & \cdots & \int a_{1M}(t)dt \\ \vdots & \ddots & \vdots \\ \int a_{M1}(t)dt & \cdots & \int a_{MM}(t)dt \end{bmatrix}$$

true only if the matrix and its integral commute

$$\mathbf{A}(\mathrm{t})\int \mathbf{A}(\tau)d\tau = \int \mathbf{A}(\tau)d\tau\ \mathbf{A}(t)$$

Computer Modeling

Ordinary differential equations continued

Function $e^{\mathbf{A}t}$

(proof from the definition)

$$e^a = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$$

$$e^a = \sum_{i=0}^{\infty}\frac{a^i}{i!}$$

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \cdots = \sum_{i=0}^{\infty}\frac{\mathbf{A}^i t^i}{i!}$$

M^2 power series – convergent

$\mathbf{A}(t)$ continuous on $b \le t \le d$, then the series converges uniformly

$$\mathbf{A} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \mathbf{0} \qquad e^{\mathbf{A}t} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \mathbf{I} \qquad e^{\mathbf{A}t} = e^t\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\frac{de^{\mathbf{A}t}}{dt} = \mathbf{A} + \mathbf{A}^2 t + \mathbf{A}^3\frac{t^2}{2!} + \cdots = \mathbf{A}e^{\mathbf{A}t}$$

1 $\mathbf{S}$ nonsingular

$$e^{\mathbf{SAS}^{-1}} = \mathbf{S}e^{\mathbf{A}}\mathbf{S}^{-1} \quad \text{(from the definition)}$$

$\mathbf{I} = \mathbf{SIS}^{-1}$ $\qquad$ $(\mathbf{SAS}^{-1}\mathbf{SAS}^{-1} = \mathbf{SA}^2\mathbf{S}^{-1})$

$e^{\mathbf{A}_1}e^{\mathbf{A}_2} = e^{\mathbf{A}_2}e^{\mathbf{A}_1}$ if $\mathbf{A}_1, \mathbf{A}_2$ commute

2 $\mathbf{A}_1, \mathbf{A}_2$ commute $(\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1)$

$$e^{\mathbf{A}_1+\mathbf{A}_2} = e^{\mathbf{A}_1}e^{\mathbf{A}_2}$$

corrolaries

$$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} \quad (\mathbf{A}t_1, \mathbf{A}t_2 \text{ commute})$$

$$\left(e^{\mathbf{A}}\right)^{-1} = e^{-\mathbf{A}} \quad (t_1 = 1,\ t_2 = -1)$$

3 $$\mathbf{A} = \begin{bmatrix}\mathbf{A}_1 & & 0\\ & \ddots & \\ 0 & & \mathbf{A}_m\end{bmatrix} \qquad e^{\mathbf{A}} = \begin{bmatrix}e^{\mathbf{A}_1} & & 0\\ & \ddots & \\ 0 & & e^{\mathbf{A}_m}\end{bmatrix}$$

because

$$\mathbf{A}^i = \begin{bmatrix}\mathbf{A}_1^i & & 0\\ & \ddots & \\ 0 & & \mathbf{A}_m^i\end{bmatrix}$$

4 $\mathbf{A}(t), \frac{d\mathbf{A}(t)}{dt}$ commute $\left(\mathbf{A}(t)\frac{d\mathbf{A}(t)}{dt} = \frac{d\mathbf{A}(t)}{dt}\mathbf{A}(t)\right)$

$$\frac{de^{\mathbf{A}(t)}}{dt} = \frac{d\mathbf{A}(t)}{dt}e^{\mathbf{A}(t)}$$

$$\det e^{\mathbf{A}} = e^{\mathrm{tr}\mathbf{A}} > 0$$

$\mathrm{tr}\,\mathbf{A} = a_{11} + \cdots + a_{MM}$

A fundamental matrix

$$e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau} \ ?$$

$$\frac{d\boldsymbol{Y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{Y}(t) \qquad \boldsymbol{Y}(0) = \mathbf{I}$$

Is it true that $\boldsymbol{Y}(t) = e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau}$?

$$e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau} = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\tau)d\tau +$$

$$+ \frac{1}{2!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^2 + \frac{1}{3!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^3 \cdots$$

$$\frac{d}{dt} e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau} = \mathbf{A}(t) + \frac{1}{2!}\mathbf{A}(t)\int_{t_0}^{t} \mathbf{A}(\tau)d\tau +$$

$$+ \frac{1}{2!}\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\,\mathbf{A}(t) + \frac{1}{3!}\mathbf{A}(t)\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^2 +$$

$$+ \frac{1}{3!}\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\,\mathbf{A}(t)\int_{t_0}^{t} \mathbf{A}(\tau)d\tau +$$

$$+ \frac{1}{3!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^2 \mathbf{A}(t) + \cdots =$$

If $\mathbf{A}(t)\int_{t_0}^{t} \mathbf{A}(\tau)d\tau = \int_{t_0}^{t} \mathbf{A}(\tau)d\tau\,\mathbf{A}(t)$ (commute)

$$\frac{d}{dt} e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau} = \mathbf{A}(t)\left[\mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\tau)d\tau + \frac{1}{2!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^2 + \frac{1}{3!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^3 \cdots\right] =$$

$$= \mathbf{A}(t)e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau}$$

In this case, $\boldsymbol{Y}(t) = e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau}$ (nonsingular) is a fundamental matrix, and it is the fundamental matrix normed at t_0, i.e.

$$\boldsymbol{\Phi}(t, t_0) = e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau}$$

Example $\mathbf{A}(t)$=a(t)**B** **B** – a constant matrix

Equations with constant coefficients

commute

$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}\boldsymbol{y}(t)$ $\quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$

$\boldsymbol{y}(t) = e^{\mathbf{A}(t-t_0)}\boldsymbol{y}_0$ $\quad \mathbf{A}\int \mathbf{A}d\tau = \int \mathbf{A}d\tau\, \mathbf{A}$

$e^{\mathbf{A}(t-t_0)}$ the normed fundamental matrix

for $t = t_0$ ($= \mathbf{I}$)

any fundamental matrix $\mathbf{Y}(t)$ satisfies

$$\frac{d\mathbf{Y}(t)}{dt} = \mathbf{A}\mathbf{Y}(t)$$

$\frac{d\boldsymbol{Y}(t)}{dt} = \mathbf{A}\boldsymbol{Y}(t)$ $\quad \boldsymbol{Y}(0) = \mathbf{I}$

$s\mathcal{L}\{\boldsymbol{Y}(t)\} - \mathbf{I} = \mathbf{A}\mathcal{L}\{\boldsymbol{Y}(t)\}$ $\quad \mathcal{L}\{\boldsymbol{Y}(t)\} = \boldsymbol{Y}(s)$

$(s\mathbf{I} - \mathbf{A})\mathcal{L}\{\boldsymbol{Y}(t)\} = \mathbf{I}$

$\mathcal{L}\{\boldsymbol{Y}(t)\} = (s\mathbf{I} - \mathbf{A})^{-1}$

[$(s\mathbf{I} - \mathbf{A})$ nonsingular generically]

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}$$

$$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}\boldsymbol{y}(t) + \boldsymbol{u}(t)$$

a general solution

$$\boldsymbol{y}(t) = e^{\mathbf{A}(t-t_0)}\boldsymbol{h} + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\boldsymbol{u}(\tau)d\tau$$

a particular solution for $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$

$$\boldsymbol{y}(t) = e^{\mathbf{A}(t-t_0)}\boldsymbol{y}_0 + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\boldsymbol{u}(\tau)d\tau$$

$s\boldsymbol{y}(s) = \mathbf{A}\boldsymbol{y}(s) + \boldsymbol{u}(s)$ $\quad (\boldsymbol{y}(t_0) = \mathbf{0})$

$(s\mathbf{I} - \mathbf{A})\, \boldsymbol{y}(s) = \boldsymbol{u}(s)$

$$\boldsymbol{y}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{u}(s)$$

$\mathbf{G}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$ the transfer matrix

$\mathbf{G}(s) = \mathcal{L}\{\boldsymbol{Y}(t)\}$

Nonsingularity of $e^{\mathbf{A}}$

$\det e^{\mathbf{A}} = e^{\mathrm{tr}\mathbf{A}}$

$\mathrm{tr}\,\mathbf{A} = a_{11} + \cdots + a_{MM}$

Sketch of a proof

$\frac{d\mathbf{Z}(t)}{dt} = \mathbf{AZ}(t), \qquad \mathbf{Z}(0) = \mathbf{I}$

Solution: $\mathbf{Z}(t) = e^{\mathbf{A}t}$ a fundamental matrix

$M = 2$

$$\begin{bmatrix} \frac{dz_{11}}{dt} & \frac{dz_{12}}{dt} \\ \frac{dz_{21}}{dt} & \frac{dz_{22}}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}z_{11} + a_{12}z_{21} & a_{11}z_{12} + a_{12}z_{22} \\ a_{21}z_{11} + a_{22}z_{21} & a_{21}z_{12} + a_{22}z_{22} \end{bmatrix}$$

$\det e^{\mathbf{A}t} = \det \mathbf{Z}(\boldsymbol{t})$

$$\frac{d}{dt}\begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} = \frac{d}{dt}(z_{11}z_{22} - z_{12}z_{21}) =$$

$$= \frac{dz_{11}}{dt}z_{22} + z_{11}\frac{dz_{22}}{dt} - \frac{dz_{12}}{dt}z_{21} - z_{12}\frac{dz_{21}}{dt} =$$

$$= \begin{vmatrix} \frac{dz_{11}}{dt} & \frac{dz_{12}}{dt} \\ z_{21} & z_{22} \end{vmatrix} + \begin{vmatrix} z_{11} & z_{12} \\ \frac{dz_{21}}{dt} & \frac{dz_{22}}{dt} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11}z_{11} + a_{12}z_{21} & a_{11}z_{12} + a_{12}z_{22} \\ z_{21} & z_{22} \end{vmatrix} +$$

$$+ \begin{vmatrix} z_{11} & z_{12} \\ a_{21}z_{11} + a_{22}z_{21} & a_{21}z_{12} + a_{22}z_{22} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11}z_{11} & a_{11}z_{12} \\ z_{21} & z_{22} \end{vmatrix} + \begin{vmatrix} z_{11} & z_{12} \\ a_{22}z_{21} & a_{22}z_{22} \end{vmatrix} =$$

$$= (a_{11} + a_{22})\det\mathbf{Z} = \mathrm{tr}\,\mathbf{A}\det\mathbf{Z}$$

$\frac{d\det\mathbf{Z}(t)}{dt} = \mathrm{tr}\,\mathbf{A}\det\mathbf{Z}(t) \qquad \det\mathbf{Z}(0) = 1$

$\det\mathbf{Z}(t) = e^{\mathrm{tr}\mathbf{A}\,t} \qquad t = 1$

$\det e^{\mathbf{A}} = e^{\mathrm{tr}\mathbf{A}}$

Nonsingularity of $e^{\mathbf{A}}$ cont.

$M = 3$

$$\begin{bmatrix} \frac{dz_{11}}{dt} & \frac{dz_{12}}{dt} & \frac{dz_{13}}{dt} \\ \frac{dz_{21}}{dt} & \frac{dz_{22}}{dt} & \frac{dz_{23}}{dt} \\ \frac{dz_{31}}{dt} & \frac{dz_{32}}{dt} & \frac{dz_{33}}{dt} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{i=1}^{3} a_{1i} z_{i1} & \sum_{i=1}^{3} a_{1i} z_{i2} & \sum_{i=1}^{3} a_{1i} z_{i3} \\ \sum_{i=1}^{3} a_{2i} z_{i1} & \sum_{i=1}^{3} a_{2i} z_{i2} & \sum_{i=1}^{3} a_{2i} z_{i3} \\ \sum_{i=1}^{3} a_{3i} z_{i1} & \sum_{i=1}^{3} a_{3i} z_{i2} & \sum_{i=1}^{3} a_{3i} z_{i3} \end{bmatrix}$$

Laplace expansion along the 1st row

$$\begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} = z_{11} \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & z_{33} \end{vmatrix} - $$
$$-z_{12} \begin{vmatrix} z_{21} & z_{13} \\ z_{31} & z_{33} \end{vmatrix} + z_{13} \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}$$

$$\frac{d}{dt} \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} = \frac{dz_{11}}{dt} \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & z_{33} \end{vmatrix} +$$
$$z_{11} \frac{d}{dt} \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & z_{33} \end{vmatrix} - \frac{dz_{12}}{dt} \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & z_{33} \end{vmatrix} -$$
$$-z_{12} \frac{d}{dt} \begin{vmatrix} z_{21} & z_{13} \\ z_{31} & z_{33} \end{vmatrix} + \frac{dz_{13}}{dt} \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix} +$$
$$+z_{13} \frac{d}{dt} \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix} = \begin{vmatrix} \frac{dz_{11}}{dt} & \frac{dz_{12}}{dt} & \frac{dz_{13}}{dt} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} +$$
$$+ \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ \frac{dz_{21}}{dt} & \frac{dz_{22}}{dt} & \frac{dz_{23}}{dt} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} + \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ \frac{dz_{31}}{dt} & \frac{dz_{32}}{dt} & \frac{dz_{33}}{dt} \end{vmatrix} =$$
$$= \begin{vmatrix} a_{11} z_{11} & a_{11} z_{12} & a_{11} z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} + \cdots =$$
$$= (a_{11} + a_{22} + a_{33}) \det \mathbf{Z} = \operatorname{tr} \mathbf{A} \det \mathbf{Z}$$

and further as before

Jordan canonical form: distinct eigenvalues

$\frac{d\mathbf{Y}(t)}{dt} = \mathbf{A}\mathbf{Y}(t)$, transformation of $\mathbf{Y}(t)$ to

$\mathbf{Z}(t) = \mathbf{S}^{-1}\mathbf{Y}(t)$ $\quad$ $\mathbf{S}$ - a nonsingular matrix

$\frac{d\mathbf{Z}(t)}{dt} = \mathbf{\Lambda}\mathbf{Z}(t)$ $\quad$ $\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$

$\mathbf{\Lambda}$ - Jordan canonical form $\quad$ $\mathbf{Z}(t) = e^{\mathbf{\Lambda}t}$

The form depends on eigenvalues of $\mathbf{A}$

$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} = \mathbf{0}$ $\quad$ the characteristic equation

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \rightarrow M$ eigenvalues λ_i

$\boldsymbol{x}_i$ - eigenvectors

<u>distinct (single) eigenvalues</u>:

columns of $\mathbf{S}$ are eigenvectors, as

$\mathbf{AS} = \mathbf{S\Lambda}$

$\mathbf{A}[\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M] = [\lambda_1\boldsymbol{s}_1, \ldots, \lambda_M\boldsymbol{s}_M]$

$(\mathbf{A} - \lambda_1\mathbf{I})\boldsymbol{s}_1 = \mathbf{0}, \ldots, (\mathbf{A} - \lambda_M\mathbf{I})\boldsymbol{s}_M = \mathbf{0}$

$$\boldsymbol{s}_i = \begin{bmatrix} s_{i1} \\ \vdots \\ s_{iM} \end{bmatrix} \neq \mathbf{0}$$

For distinct eigenvalues $\lambda_i \neq \lambda_j$ for $i \neq j$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_M \end{bmatrix} \quad e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_M t} \end{bmatrix}$$

$$\mathbf{Y}(t) = \mathbf{SZ}(t) = \begin{bmatrix} s_{11}e^{\lambda_1 t} & \cdots & s_{1M}e^{\lambda_M t} \\ \vdots & \ddots & \vdots \\ s_{M1}e^{\lambda_1 t} & \cdots & s_{MM}e^{\lambda_M t} \end{bmatrix}$$

$\mathbf{Y}(t)$ - a fundamental matrix (**Example (£)**)

a general solution of a nonhomogeneous equation for distinct eigenvalues

$$\boldsymbol{y}^0(t) = \mathbf{Y}(t)\boldsymbol{h} = \sum_{i=1}^{M} h_i\boldsymbol{s}_i e^{\lambda_i} = \sum_{i=1}^{M} \boldsymbol{c}_i e^{\lambda_i}$$

An example (1)

$$\frac{dy_1(t)}{dt} = 5y_1(t) + 4y_2(t)$$
$$\frac{dy_2(t)}{dt} = 4y_1(t) + 5y_2(t)$$

the characteristic equation

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 9 = 0$$

$\lambda_1 = 1, \quad \lambda_2 = 9$

an eigenvector for $\lambda_1 = 1$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} = \mathbf{0}, \; s_{11} = 1, s_{21} = -1$$

an eigenvector for $\lambda_2 = 9$

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} = \mathbf{0}, \; s_{12} = 1, s_{22} = 1$$

$$W(t) = \begin{vmatrix} e^t & e^{9t} \\ -e^t & e^{9t} \end{vmatrix} = 2e^{10t} > 0$$

a fundamental matrix

$$Y(t) = \begin{bmatrix} e^t & e^{9t} \\ -e^t & e^{9t} \end{bmatrix}$$

a general solution

$y_1(t) = h_1 e^t + h_2 e^{9t}$

$y_2(t) = -h_1 e^t + h_2 e^{9t}$

checking

L_1: $h_1 e^t + 9h_2 e^{9t}$

P_1: $5(h_1 e^t + h_2 e^{9t}) + 4(-h_1 e^t + h_2 e^{9t}) = L_1$

L_2: $-h_1 e^t + 9h_2 e^{9t}$

P_2: $4(h_1 e^t + h_2 e^{9t}) + 5(-h_1 e^t + h_2 e^{9t}) = L_2$

An example (2)

$$\frac{dy_1(t)}{dt} = y_1(t) - y_2(t)$$
$$\frac{dy_2(t)}{dt} = y_1(t) + y_2(t)$$

the characteristic equation

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i$$

an eigenvector for $\lambda_1 = 1 + i$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} = \mathbf{0}, \; s_{11} = i, s_{21} = 1$$

an eigenvector for $\lambda_2 = 1 - i$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} = \mathbf{0}, \; s_{12} = i, s_{22} = -1$$

$$y_{11}(t) = ie^{(i+1)t} = -e^t \sin t + ie^t \cos t$$
$$y_{21}(t) = e^{(i+1)t} = e^t \cos t + ie^t \sin t$$
$$y_{12}(t) = ie^{(i-1)t} = e^t \sin t + ie^t \cos t$$
$$y_{22}(t) = -e^{(i-1)t} = -e^t \cos t + ie^t \sin t$$

$$W(t) = \begin{vmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{vmatrix} = e^{2t} > 0$$

a fundamental matrix

$$Y(t) = \begin{bmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{bmatrix}$$

a general solution

$$y_1(t) = h_1 e^t \cos t - h_2 e^t \sin t$$
$$y_2(t) = h_1 e^t \sin t + h_2 e^t \cos t$$

Jordan canonical form: multiple eigenvalues

$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{\Lambda}_L \end{bmatrix}$$

L – number of different eigenvalues

$$\mathbf{\Lambda}_l = \begin{bmatrix} \mathbf{\Lambda}_{l1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{\Lambda}_{lk_l} \end{bmatrix}$$

k_l - multiplicity of the l-th eigenvalue

$$\mathbf{\Lambda}_{lj} = \begin{bmatrix} \lambda_l & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_l & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_l & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda_l \end{bmatrix}$$

a Jordan block

$$\mathbf{\Lambda}_{lj} = \lambda_l \mathbf{I} + \mathbf{F}_{lj} \qquad \mathbf{F}_{lj} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\mathbf{\Lambda}_{11}t} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{\mathbf{\Lambda}_{Lk_L}t} \end{bmatrix}$$

$$e^{\mathbf{\Lambda}_{lj}t} = e^{(\lambda_l \mathbf{I} + \mathbf{F}_{lj})t} = e^{\lambda_l t} e^{\mathbf{F}_{lj}t}$$

$\mathbf{I}$ and $\mathbf{F}_{lj}$ commute

$$e^{\mathbf{F}_{lj}t} = \mathbf{I} + \mathbf{F}_{lj}t + \frac{1}{2!}\mathbf{F}_{lj}^2 t^2 + \cdots$$

assume $k_l = 3$

$$\mathbf{F}_{lj} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{F}_{lj}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{F}_{lj}^3 = \mathbf{0}$$

$$e^{\mathbf{\Lambda}_{lj}t} = e^{\lambda_l t}(\mathbf{I} + \mathbf{F}_{lj}t + \frac{1}{2!}\mathbf{F}_{lj}^2 t^2) = e^{\lambda_l t} \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2!}t^2 & t & 1 \end{bmatrix}$$

in general

$$\boldsymbol{y}(t) = \sum_{l=1}^{L} \sum_{j=0}^{k_l - 1} t^j e^{\lambda_l t} \mathbf{Z}_{lj} \boldsymbol{h}$$

$\mathbf{Z}_{lj}$ depends only on $\mathbf{A}$, not on t

Jordan canonical form: multiple eigenvalues

$$\boldsymbol{y}(t) = \sum_{l=1}^{L} \sum_{j=0}^{k_l-1} t^j e^{\lambda_l t} \mathbf{Z}_{lj} \boldsymbol{h} =$$

$$= e^{\lambda_1 t}\, \mathbf{c}_{10} + t e^{\lambda_1 t} \mathbf{c}_{11} + \cdots + t^{k_1-1} e^{\lambda_1 t} \mathbf{c}_{1,k_1-1} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ e^{\lambda_L t} \mathbf{c}_{L0} + t e^{\lambda_L t} \mathbf{c}_{L1} + \cdots + t^{k_L-1} e^{\lambda_L t} \mathbf{c}_{L,k_L-1}$$

Jordan decomposition is numericaly unstable

Example

$$\mathbf{A} = \begin{bmatrix} 1 & \varepsilon \\ 1 & 1 \end{bmatrix}$$

For $\varepsilon = 0$ the Jordan form

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

For $\varepsilon > 0$

$$\begin{vmatrix} 1-\lambda & \varepsilon \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 - \varepsilon = 0$$

$$\lambda_1 = 1 - \sqrt{\varepsilon}, \qquad \lambda_2 = 1 + \sqrt{\varepsilon}$$

$$\begin{bmatrix} \sqrt{\varepsilon} & \varepsilon \\ 1 & \sqrt{\varepsilon} \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} = 0 \quad \begin{matrix} s_{11} = -\sqrt{\varepsilon} \\ s_{21} = 1 \end{matrix}$$

$$\begin{bmatrix} -\sqrt{\varepsilon} & \varepsilon \\ 1 & -\sqrt{\varepsilon} \end{bmatrix} \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} = 0 \quad \begin{matrix} s_{12} = 1 \\ s_{22} = \sqrt{\varepsilon} \end{matrix}$$

$$\mathbf{S} = \begin{bmatrix} -\sqrt{\varepsilon} & 1 \\ 1 & \sqrt{\varepsilon} \end{bmatrix} \qquad \mathbf{S}^{-1} = \begin{bmatrix} -\frac{1}{2\sqrt{\varepsilon}} & \frac{1}{2} \\ \frac{1}{2\sqrt{\varepsilon}} & \frac{1}{2} \end{bmatrix}$$

$$\boldsymbol{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} -\frac{1}{2\sqrt{\varepsilon}} & \frac{1}{2} \\ \frac{1}{2\sqrt{\varepsilon}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \varepsilon \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{\varepsilon} & 1 \\ 1 & \sqrt{\varepsilon} \end{bmatrix} =$$

$$= \begin{bmatrix} 1-\sqrt{\varepsilon} & 0 \\ 0 & 1+\sqrt{\varepsilon} \end{bmatrix}$$

Calculation of $e^{\mathbf{A}t}$

from the definition (1)

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots = \sum_{i=0}^{\infty} \frac{\mathbf{A}^i t^i}{i!}$$

1. $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad e^{\mathbf{A}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
2. $\mathbf{A} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ $\quad e^{\mathbf{A}t} = \begin{bmatrix} 1 & bt \\ 0 & 1 \end{bmatrix}$
3. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad e^{\mathbf{A}t} = e^{t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
4. $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ $\quad e^{\mathbf{A}t} = e^{at}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
5. $\mathbf{A} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ $\quad e^{\mathbf{A}t} = e^{at}\begin{bmatrix} 1 & bt \\ 0 & 1 \end{bmatrix}$

Hints:

2. $\mathbf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
3. $\mathbf{A}^i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
4. $\mathbf{A}^i = \begin{bmatrix} a^i & 0 \\ 0 & a^i \end{bmatrix}$
5. $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ (commute)

$e^{\mathbf{A}_1+\mathbf{A}_2} = e^{\mathbf{A}_1}e^{\mathbf{A}_2}$ $\quad (\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1)$

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix}$$

Calculation of $e^{\mathbf{A}t}$
from the definition (2)

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots = \sum_{i=0}^{\infty} \frac{\mathbf{A}^i t^i}{i!}$$

6. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad e^{\mathbf{A}t} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{A}^i = \begin{cases} \mathbf{A} & \text{for } i \text{ odd} \\ \mathbf{I} & \text{for } i \text{ even} \end{cases}$$

$$\sum_{i=0}^{\infty} \frac{\mathbf{A}^i t^i}{i!} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{I}t^2}{2!} + \frac{\mathbf{A}t^3}{2!} + \cdots =$$

$$= \begin{bmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots & t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \\ t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) & \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) & \frac{1}{2}(e^t + e^{-t}) \end{bmatrix} =$$

$$= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

7. $\mathbf{B} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$ $\quad e^{\mathbf{B}t} = \begin{bmatrix} \cosh bt & \sinh bt \\ \sinh bt & \cosh bt \end{bmatrix}$

$$\mathbf{B} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = b\mathbf{A}$$

$$\mathbf{B}^2 = b^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = b^2\mathbf{I} \qquad \mathbf{B}^3 = b^3\mathbf{A}$$

$$\mathbf{B}^i = \begin{cases} b^i\mathbf{A} & \text{for } i \text{ odd} \\ b^i\mathbf{I} & \text{for } i \text{ even} \end{cases}$$

$$\sum_{i=0}^{\infty} \frac{\mathbf{B}^i t^i}{i!} = \mathbf{I} + \mathbf{A}bt + \frac{\mathbf{I}(bt)^2}{2!} + \frac{\mathbf{A}(bt)^3}{2!} + \cdots =$$

$$= \begin{bmatrix} 1 + \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} + \cdots & bt + \frac{(bt)^3}{3!} + \frac{(bt)^5}{5!} + \cdots \\ bt + \frac{(bt)^3}{3!} + \frac{(bt)^5}{5!} + \cdots & 1 + \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(e^{bt} + e^{-bt}) & \frac{1}{2}(e^{bt} - e^{-bt}) \\ \frac{1}{2}(e^{bt} - e^{-bt}) & \frac{1}{2}(e^{bt} + e^{-bt}) \end{bmatrix} =$$

$$= \begin{bmatrix} \cosh bt & \sinh bt \\ \sinh bt & \cosh bt \end{bmatrix}$$

$$e^t + e^{-t} = 1 + t + \frac{t^2}{2!} + \cdots 1 - t + \frac{t^2}{2!} + \cdots$$

Calculation of $e^{\mathbf{A}t}$

from the Cayley-Hamilton theorem

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^M + \alpha_{M-1}\lambda^{M-1} + \cdots + \alpha_0 = 0$$

Matrix $\mathbf{A}$ satisfies its characteristic equation

$$\mathbf{A}^M + \alpha_{M-1}\mathbf{A}^{M-1} + \cdots + \alpha_0\mathbf{I} = \mathbf{0}$$

Minimal polynomial

$$\mathbf{A}^J + \kappa_{M-1}\mathbf{A}^{J-1} + \cdots + \kappa_0\mathbf{I} = \mathbf{0} \qquad J \le M$$

The minimal polynomial is unique,
λ_i are the roots of the minimal polynomial,
but possibly of lower multiplicity.

Consider differential equation

$$\frac{d^J f(t)}{dt^J} + \kappa_{M-1}\frac{d^{J-1} f(t)}{dt^{J-1}} + \cdots + \kappa_0 f(t) = 0$$

Let a set of functions

$$f_p(t), \qquad p = 0,1,\dots,J-1$$

be the particular solutions satisfying the initial conditions

$$f_p^{(r)}(0) = \begin{cases} 0 & \text{for} \quad r \ne p \\ 1 & \text{for} \quad r = p \end{cases}$$

Then

$$\boxed{e^{\mathbf{A}t} = f_{J-1}(t)\mathbf{A}^{J-1} + \cdots + f_0(t)\mathbf{I}}$$

An example

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) =$$

$$= \begin{vmatrix} 1-\lambda & 5 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$$

$\lambda = 1$ – the triple eigenvalue

$(\mathbf{I} - \mathbf{A})^3 = \mathbf{0}$ from the Cayley-Hamilton theorem

$$\begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The minimal polynomial ($J = 2$)

$(\lambda - 1)^2 = \lambda^2 - 2\lambda + 1 = 0$

$$\frac{d^2 f(t)}{dt^2} - 2\frac{df(t)}{dt} + f(t) = 0$$

solutions: $f_1(t) = e^t \quad f_2(t) = te^t$

$g_1(t) = f(t), \; g_2(t) = f'(t), \; \frac{d}{dt}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$

$$W(t) = \begin{vmatrix} f_{11}(t) & f_{21}(t) \\ f'_{11}(t) & f'_{22}(t) \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} > 0$$

a general solution $f(t) = \gamma_1 e^t + \gamma_2 te^t$

the particular solutions

$f_0(0) = 1, \; f_0^{(1)}(0) = 0\!: \; f_0(t) = e^t - te^t$

$f_1(0) = 0, \; f_1^{(1)}(0) = 1\!: \; f_1(t) = te^t$

$$e^{\mathbf{A}t} = te^t \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + (e^t - te^t)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 5te^t & 0 \\ 0 & e^t & 0 \\ 0 & te^t & e^t \end{bmatrix}$$

$$e^{\mathbf{A}t} = f_{J-1}(t)\mathbf{A}^{J-1} + \cdots + f_0(t)\mathbf{I}$$

Calculation of $e^{\mathbf{A}t}$

from the Laplace transform

$e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}$

[$s \neq s_i$ - any eigenvalue]

<u>Lemma</u>

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathbf{B}(s)}{d(s)}$$

where

$d(s) = \det(s\mathbf{I} - \mathbf{A}) = s^M + d_1 s^{M-1} + \cdots$
$+ d_{M-1} s + d_M$

$\mathbf{B}(s) = s^{M-1}\mathbf{B}_0 + s^{M-2}\mathbf{B}_1 + \cdots + s\mathbf{B}_{M-2} + \mathbf{B}_{M-1}$

<u>Proof</u>: From the Cramer's rule, the (i, j) element of $(s\mathbf{I} - \mathbf{A})^{-1}$ is equal to $P_{ij}(s)/d(s)$, where $P_{ij}(s)$ is the cofactor of the (i, j) element. The cofactor is equal to $(-1)^{i+j}$ times the determinant of a $(M-1) \times (M-1)$ matrix.

<u>Formulae for matrices $\mathbf{B}_i$</u>

$\mathbf{B}_0 = \mathbf{I}$ $\quad d_1 = -\mathrm{tr}\,\mathbf{A}$

$\mathbf{B}_1 = \mathbf{B}_0\mathbf{A} + d_1\mathbf{I}$ $\quad d_2 = -\frac{1}{2}\mathrm{tr}\,(\mathbf{B}_1\mathbf{A})$

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$\mathbf{B}_k = \mathbf{B}_{k-1}\mathbf{A} + d_k\mathbf{I}$ $\quad d_k = -\frac{1}{k}\mathrm{tr}\,(\mathbf{B}_{k-1}\mathbf{A})$

… … … … … … … … … … … … … … … … … … … …

$\mathbf{B}_{M-1} = \mathbf{B}_{M-2}\mathbf{A} + d_M\mathbf{I}$ $\quad d_{M-1} = -\frac{1}{M-1}\mathrm{tr}\,(\mathbf{B}_{M-2}\mathbf{A})$

$d_M = -\frac{1}{M}\mathrm{tr}\,(\mathbf{B}_{M-1}\mathbf{A})$

<u>Comment</u>

The elements of the $(s\mathbf{I} - \mathbf{A})^{-1}$ matrix are rational functions of the variable s. It may happen that some or even all elemements have common factors both in the nominator and the denominator.

<u>Proof</u> L.A.Zadeh, C.A.Desoer (1963) *Linear System Theory*. McGraw-Hill, New York.

An example

$$\frac{dy_1(t)}{dt} = -2y_1(t) + y_2(t)$$
$$\frac{dy_2(t)}{dt} = 2y_1(t) - 3y_2(t)$$

$$\frac{d\boldsymbol{y}(t)}{dt} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \boldsymbol{y}(t)$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s+2 & -1 \\ -2 & s+3 \end{bmatrix}$$

$$\begin{vmatrix} s+2 & -1 \\ -2 & s+3 \end{vmatrix} = s^2 + 5s + 4 = (s+1)(s+4)$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+4)} & \frac{1}{(s+1)(s+4)} \\ \frac{2}{(s+1)(s+4)} & \frac{s+2}{(s+1)(s+4)} \end{bmatrix}$$

$$\frac{s+3}{(s+1)(s+4)} = \frac{2}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4}$$
$$\frac{2}{(s+1)(s+4)} = \frac{2}{3}\frac{1}{s+1} - \frac{2}{3}\frac{1}{s+4}$$
$$\frac{1}{(s+1)(s+4)} = \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}$$
$$\frac{s+2}{(s+1)(s+4)} = \frac{1}{3}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s+4}$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-\mathbf{1}}\{(s\mathbf{I} - \mathbf{A})^{-1}\} =$$
$$= \frac{1}{3}\begin{bmatrix} 2e^{-t} + e^{-4t} & e^{-t} + 2e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} - e^{-4t} \end{bmatrix}$$

Equations with periodic coefficients

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}(t)\mathbf{y}(t) \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0$$

$\mathbf{A}(t)$ –piecewise continuous periodic function of t with a period T $\quad M \times M$

$$\mathbf{A}(t+T) = \mathbf{A}(t)$$

The solution

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t,0)\boldsymbol{y}_0 \overset{\text{def}}{=} \boldsymbol{\Phi}(t)\boldsymbol{y}_0$$

$\boldsymbol{\Phi}(t)$ - the normed fundamental matrix

$$\frac{d\boldsymbol{\Phi}(t)}{dt} = \mathbf{A}(t)\boldsymbol{\Phi}(t) \qquad \boldsymbol{\Phi}(0) = \mathbf{I}$$

$$\frac{d\boldsymbol{\Phi}(t+T)}{dt} = \mathbf{A}(t+T)\boldsymbol{\Phi}(t+T) =$$

$$= \mathbf{A}(t)\boldsymbol{\Phi}(t+T)$$

$\boldsymbol{\Phi}(t+T)$ - a fundamental matrix

$\boldsymbol{\Phi}(T)$ $(t = 0)$ nonsingular

$$\boldsymbol{\Psi}(t) \overset{\text{def}}{=} \boldsymbol{\Phi}(t+T)\boldsymbol{\Phi}^{-1}(T)$$

$$\frac{d\boldsymbol{\Psi}(t)}{dt} = \mathbf{A}(t)\boldsymbol{\Psi}(t) \qquad \boldsymbol{\Psi}(0) = \mathbf{I}$$

$\boldsymbol{\Psi}(t)$ - the normed fundamental matrix

$$\boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)$$

$$\boldsymbol{\Phi}(t+T) = \boldsymbol{\Phi}(t)\,\boldsymbol{\Phi}(T)$$

$$\boldsymbol{\Phi}(t+nT) = \boldsymbol{\Phi}(t)\,\boldsymbol{\Phi}^n(T)$$

$$\boldsymbol{y}(t+nT) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^n(T)\boldsymbol{y}_0$$

$$\boldsymbol{y}(nT) = \boldsymbol{\Phi}^n(T)\boldsymbol{y}_0$$

Equations with periodic coefficients

consider an equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{B}\mathbf{x}(t) \qquad \boldsymbol{x}(0) = \boldsymbol{y}_0$$

$\mathbf{B}$ – a constant matrix

$\mathbf{x}(t) = e^{\mathbf{B}t}\boldsymbol{y}_0$

require $\quad \boldsymbol{y}(nT) = \mathbf{x}(nT)$

$\boldsymbol{\Phi}^n(T)\boldsymbol{y}_0 = \left(e^{\mathbf{B}T}\right)^n \boldsymbol{y}_0$

$\mathbf{C} = [\boldsymbol{y}_0^1, \boldsymbol{y}_0^2, \cdots, \boldsymbol{y}_0^M]$ - chosen to be nonsingular

$\boldsymbol{\Phi}^n(T)\mathbf{C} = \left(e^{\mathbf{B}T}\right)^n \mathbf{C} \qquad \times \mathbf{C}^{-1}$

$n = 1$

$\boldsymbol{\Phi}(T) = e^{\mathbf{B}T}$

$\boldsymbol{\Phi}(t) = \boldsymbol{\Phi}(t)e^{-\mathbf{B}t}e^{\mathbf{B}t} \overset{\text{def}}{=} \mathbf{P}(t)e^{\mathbf{B}t}$

A fundamental matrix in the Floquet normal form

$\boxed{\boldsymbol{y}(t) = \mathbf{P}(t)e^{\mathbf{B}t}\boldsymbol{y}_0} \quad \mathbf{P}(0) = \boldsymbol{\Phi}(0) = \mathbf{I}$

The Floquet solution

$\mathbf{P}(t)$ periodic, as

$\mathbf{P}(t+T) = \boldsymbol{\Phi}(t+T)e^{-\mathbf{B}(t+T)} =$

$\boldsymbol{\Phi}(t)\,\boldsymbol{\Phi}(T)e^{-\mathbf{B}t}e^{-\mathbf{B}T} =$ (exponents commute)

$\boldsymbol{\Phi}(t)e^{\mathbf{B}T}e^{-\mathbf{B}T}e^{-\mathbf{B}t} =$

$= \boldsymbol{\Phi}(t)e^{-\mathbf{B}t} = \mathbf{P}(t)$

$\mathbf{P}(t)$ continuous and periodic, thus bounded

Thank you!