Hard and Soft Sub-Time-Optimal Robust Controllers

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Abstract: In many applicational tasks of motion control – fundamental for research in robotics – problems associated with uncertain and/or varying load (a mass or moment of inertia) can present a substantial difficulty during the synthesis of practical controlling systems. The random concept, where the load has been treated as a stochastic process, is presented in this paper. As a result, through a generalization of the classic switching curve occurring in the time-optimal approach, two control structures have been investigated: the hard, defined on the basis of the rules of the statistical decision theory, and also the soft, which additionally allows the elimination of rapid changes in control values. The methodology proposed here may be easily adopted for other elements commonly found in mechanical systems, e.g. parameters of drive or motion resistance, giving the sub-time-optimal controlling structures that provide many advantages, especially with respect to robustness.

Keywords: control systems design, mechanical systems, uncertain mass, random approach, suboptimal structures, robustness.

1. INTRODUCTION

One of the main sources of uncertainty in models of mechanical systems – a basis for research in the field of robotics (Bryson and Ho, 1975; Chandresekharan, 1996; Khalil, 2005) – is the load, or, to be more precise, the value of a mass or a moment of inertia. In practice this value may be given with only the degree of precision allowed by measurement errors. Moreover, in many applications this value is not subject to measurement at all, but rather is grossly estimated on the basis of an assumed value. In still other cases, the load may vary, together with the consumption of substances used in the technological process.

In the present paper this problem has been solved by the introduction of the random factor; namely, a load will be treated as the realization of a stochastic process. The introduction of a random factor makes it possible to take into account errors in the identification of mass, whereas the fluctuations of the particular realizations describe its changes, including also those of a discontinuing nature.

The above concept allowed to propose new types of control structures that take into account an uncertain load, without the undue complication of a control law. The first of these – termed “hard” – has been based on principals of statistical decision theory. Next, to eliminate frequent switchings occurring on the sliding trajectories, which in mechanical systems have a negative impact on the actuator life and may excite vibrations in elastic transmissions, the second, “soft” structure has also been investigated and presented below. Empirical tests have confirmed the satisfactory performance of the structures proposed, indicating a considerable number of advantages, especially with respect to robustness.

The material of this paper was presented in details in the article (Kulczycki et al, 2004). It is a development of research concerning robust control of mechanical systems, primarily published among others in the papers (Kulczycki, 1996a,b,c, 2000).

2. THEORETICAL RESULTS

The random approach for the control task presented here has been based on the concept of an almost certain time-optimal control. This is defined as a stochastic process such that almost all its realizations are controls which, for proper deterministic systems obtained by fixing the random factor, bring the state of the system to the target set in a minimal and finite time. The almost certain time-optimal control is unique if every time-optimal control is a process stochastically equivalent to it. This notion was introduced and developed by Kulczycki (1996a,b). Similarly, an almost certain solution of a random differential equation means such a stochastic process that almost all its realizations are solutions of proper deterministic equations obtained for a fixed random factor. The almost certain solution is unique if every almost certain solution is a process stochastically equivalent to it. The solution of a deterministic differential equation will be considered below in the Caratheodory sense, i.e. as a function which is absolutely continuous at every compact subinterval of its time domain and fulfils the differential equation almost everywhere; for details see (Kulczycki, 1996c).

Consider a mechanical system with a single degree of freedom, whose dynamics are described by the second law of Newtonian mechanics

$$m\ddot{s}(t)=u(t) \quad , \tag{1}$$

where m, s, u mean the load (mass or moment of inertia), position (linear or angular), and control (force or moment), respectively. If the parameter m is treated as a realization of a stochastic process *M*, then denoting by $\omega\in\Omega$ a random factor, and by X_1, X_2, U real stochastic processes which represent the position, velocity and control respectively, the dynamics of the system under consideration can now be described by the following random differential equation:

$$\dot{X}_1(\omega,t)=X_2(\omega,t) \tag{2}$$

$$\dot{X}_2(\omega,t)=\frac{1}{M(\omega,t)}U(\omega,t) \tag{3}$$

with the initial condition

$$\begin{bmatrix} X_1(\omega,t_0) \\ X_2(\omega,t_0) \end{bmatrix}=x_0 \quad \text{for almost all} \quad \omega\in\Omega \quad , \tag{4}$$

given these assumptions

(A1) $t_0\in\mathsf{R}$, $T=[t_0,\infty)$;

(A2) $x_0=[x_{01},x_{02}]^{\mathrm{T}}\in\mathsf{R}^2$, $x_f=[x_{f1},x_{f2}]^{\mathrm{T}}\in\mathsf{R}^2$ constitute initial and target states, respectively;

(A3) the values of admissible controls are limited to the interval [–1,1];

(A4) (Ω,Σ,P) denotes a complete probability space;

(A5) *M* is a real stochastic process with almost all realizations being piecewise continuous and satisfying the boundary condition $M(\omega,t)\in[m_-,m_+]$ for $t\in T$, where $0<m_-\le m_+$.

Introduce also the following subdivision of the state space R^2 into the disjoint sets R_+, R_-, Q_+, Q_-, $\{x_f\}$; see Fig. 1. Specifically, let K_{+-}, K_{++} denote sets of all states which can be brought to the target by the control $U\equiv+1$, if $M\equiv m_-$ or $M\equiv m_+$, respectively; analogously K_{--} and K_{-+} for $U\equiv-1$, if $M\equiv m_-$ or $M\equiv m_+$. Moreover, let:

$$Q_+=\{[x_1,x_2]^{\mathrm{T}}\in\mathsf{R}^2 \text{ such that there exist } [x_1',x_2]^{\mathrm{T}}\in K_{+-} \text{ and } [x_1'',x_2]^{\mathrm{T}}\in K_{++} \text{ with } x_1'\le x_1\le x_1'' \text{ or } x_1''\le x_1\le x_1'\ \} \tag{5}$$

$$Q_-=\{[x_1,x_2]^{\mathrm{T}}\in\mathsf{R}^2 \text{ such that there exist } [x_1',x_2]^{\mathrm{T}}\in K_{-+} \text{ and } [x_1'',x_2]^{\mathrm{T}}\in K_{--} \text{ with } x_1'\le x_1\le x_1'' \text{ or } x_1''\le x_1\le x_1'\ \} \tag{6}$$

$$R_+=\{[x_1,x_2]^{\mathrm{T}}\in\mathsf{R}^2\setminus Q \text{ such that there exists } [x_1',x_2]^{\mathrm{T}}\in Q \text{ with } x_1<x_1'\ \} \tag{7}$$

$$R_-=\{[x_1,x_2]^{\mathrm{T}}\in\mathsf{R}^2\setminus Q \text{ such that there exists } [x_1',x_2]^{\mathrm{T}}\in Q \text{ with } x_1'<x_1\ \} \quad , \tag{8}$$

where $Q=Q_+\cup\{x_f\}\cup Q_-$. Therefore, the sets K_{+-}, K_{++} represent all those states which can be brought to the target by the control $+1$, at the minimum and maximum possible values of a mass. The set Q_+ contains intermediate points. The sets K_{-+}, K_{--}, and Q_- may be interpreted analogously for the control -1. Note also that K_{+-} and K_{++} belong to Q_+ as K_{-+} and K_{--} belong to Q_-. See also Fig. 1.

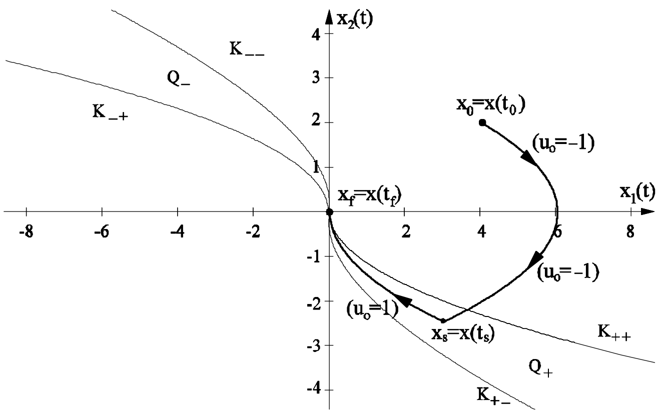

Fig. 1. Illustration of the notations and Theorem.

Theorem (Kulczycki et al, 2004)
For a dynamic system described by random differential equation (2)-(4), under assumptions (A1)-(A5), there exists a unique almost certain time-optimal control U_0, generating a unique almost certain solution $X=[X_1,X_2]^{\mathrm{T}}$, where with probability 1:

(T1) if $x_0\in R_-$, the function $U_0(\omega,\cdot)$ takes on the value -1 for $t\in[t_0,t_s(\omega))$ and $+1$ for $t\in[t_s(\omega),t_f(\omega)]$, where $t_0<t_s(\omega)<t_f(\omega)<\infty$ and $X(\omega,t)\in Q_+$ for $t\in[t_s(\omega),t_f(\omega))$; (for interpretation see Fig. 1);

(T2) if $x_0\in R_+$, the function $U_0(\omega,\cdot)$ takes on the value $+1$ for $t\in[t_0,t_s(\omega))$ and -1 for $t\in[t_s(\omega),t_f(\omega)]$, where $t_0<t_s(\omega)<t_f(\omega)<\infty$ and $X(\omega,t)\in Q_-$ for $t\in[t_s(\omega),t_f(\omega))$;

(T3) if $x_0\in Q_-$, the function $U_0(\omega,\cdot)$ takes on the form described above in points (T1) or (T2) or takes on the value -1 for $t\in[t_0(\omega),t_f(\omega)]$, where $t_0<t_f(\omega)<\infty$ and $X(t)\in Q_-$ for $t\in[t_0(\omega),t_f(\omega))$;

(T4) if $x_0\in Q_+$, the function $U_0(\omega,\cdot)$ takes on the form described above in points (T1) or (T2) or takes on the value $+1$ for $t\in[t_0(\omega),t_f(\omega)]$, where $t_0<t_f(\omega)<\infty$ and $X(t)\in Q_+$ for $t\in[t_0(\omega),t_f(\omega))$.

The functions $t_s : \Omega \to \mathrm{R}$ and $t_f : \Omega \to \mathrm{R}$ introduced above, representing the time of the changes in the value of the function $U_0(\omega, \cdot)$ and the time to reach the target by the solution $X(\omega, \cdot)$, respectively, are random variables. ■

The change of sign in the particular realizations of the control U_0 (switching of the control) can occur only when the system state belongs to the set Q. For this reason it will be called a switching region. Finally: the switching curve γ familiar from the classic case of the time-optimal point-to-point transfer of the fixed mass m (Athans, Falb, 1966, Chapter 7.2), has been generalized by the above to the switching region Q ($\gamma = Q$ when $m_- = m_+ = m$).

3. APPLICATIONAL CONCLUSIONS: SUBOPTIMAL CONTROL STRUCTURES

Besides specific cases, the direct implementation of a system generating the almost certain time-optimal control encounters difficulties because of its dependence on the random factor, in fact unknown *a priori*. However, thanks to the results of Theorem given in Section 2, the presented material constitutes a useful basis for the creation of suboptimal control laws, in which such a dependence is removed.

3.1 Hard Structure

The following concept will be based on the form of differential equation (3). Namely, after its integration on both sides, one may observe that the impact of the particular realizations of the stochastic process M can be estimated by using their mean-values over any interval of time in which no special event – for example control switching – occurs. To obtain a suboptimal controller, consider a particular case of the probability measure P connected with the process M (see Assumptions (A4)-(A5)) which is concentrated on constant realizations (interpreted as the average values). If the value of these constant realizations is known and equal to m, then with the notation of Theorem presented in the previous section, $m_- = m_+ = m$, therefore, $K_{+-} = K_{++}$ and $K_{-+} = K_{--}$, hence the switching region Q is confined to the switching curve whose shape is dependent on the value of the parameter m. Denote as $\hat{m}$ its estimate used in the feedback control law; therefore, it can be interpreted as an (indefinite) knowledge about the parameter m needed for the purpose of the synthesis of the feedback controller equations.

The analysis of sensitivity to the error of the estimation of the parameter m by the value $\hat{m}$ will be presented below.

The case where the second coordinate of the target state is equal to zero, i.e. with $x_{f2} = 0$, will be considered first. If $\hat{m} = m$, the control is time-optimal; the state of the system is brought to the switching curve, and being permanently included in this curve hereafter, it reaches the target in a minimal and finite time. When $\hat{m} < m$; as a result of its having oscillations around the target, over-regulations occur in the system; the target is reached in a finite time. If $\hat{m} > m$, after the switching curve is crossed, sliding trajectories appear in the system; here, too, the target is reached in a finite time. In both of the last two cases, i.e. with $\hat{m} \neq m$, the time to reach the target state increases from the optimal more or less proportionally to the difference between the values $\hat{m}$ and m.

The remaining case, $x_{f2} \neq 0$, will now be presented. If $\hat{m} = m$, the control is time-optimal, and the phenomena are identical as before for $x_{f2} = 0$. When $\hat{m} < m$, the trajectories occurring in the system generate limit cycles; the target is not reached. Finally if $\hat{m} > m$, even though some of the trajectories temporarily diverge from the switching curve in the part between the axis x_1 and the target state, ultimately the target is reached in a finite time; sliding trajectories exist on the switching curve; the time to reach the target increases in tandem with the growth in the difference $\hat{m} - m$.

Based on the sensitivity analysis presented above, some elements of statistical decision theory will be applied to obtain the optimal value of the estimator $\hat{m}$ needed for the purpose of the synthesis of the feedback controller equations. The basic task of statistical decision theory is the optimal selection of one element from among all possible decisions on the sole basis of probabilistic information about the state of nature (reality), especially when its actual state is unknown. In the problem considered here, the real value of the parameter m is treated as an unknown state of reality, while the fixed value of the estimator $\hat{m}$ constitutes a decision. The loss function l is required, which value $l(\hat{m}, m)$ is interpreted as losses resulting from making the decision $\hat{m}$ when hypothetically the value m occurs in reality. Two basic procedures are commonly used: the "flexible" Bayes rule minimizes the expected value of losses, whereas the "radical" minimax rule minimizes the greatest possible loss that may occur after a given decision is made. For details see (Berger, 1980).

Assume – according to the results of the sensitivity analysis – that the loss function is described in the linear and nonsymmetrical form:

$$l(\hat{m}, m) = \begin{cases} -p(\hat{m} - m) & \text{if} \quad \hat{m} - m < 0 \\ 0 & \text{if} \quad \hat{m} - m = 0 \\ q(\hat{m} - m) & \text{if} \quad \hat{m} - m > 0 \end{cases}, \tag{9}$$

where $p, q \in \mathrm{R}_+ \cup \{\infty\}$, but only one of them can be infinite. Suppose – in reference to Assumption (A5) – that the random variable characterizing the distribution of the mass m has a support of the form $[m_-, m_+]$ such that $[m_-, m_+] \subset (0, \infty)$.

It is readily shown that if $p = \infty$, i.e. with infinite values of loss function (9) for $\hat{m} < m$, the minimax decision is realized by

$$\hat{m} = m_+ \quad . \tag{10}$$

In turn, the Bayes decision with the positive numbers p and q, is given as a solution of the following equation with the argument $\hat{m}$:

$$F(\hat{m}) = \frac{p}{p+q} \quad , \tag{11}$$

where F denotes the distribution function of the random variable characterizing the mass m. This solution is unique thanks to the connectivity of its support. The practical algorithm to solve equation (11) is presented in the paper (Kulczycki, Charytanowicz, 2008) also in the presence of conditional factors. Detailed considerations can be found in the publications (Kulczycki, 2005, 2008).

The results given by formulas (10) and (11) will be applied below.

Once again the case $x_{f2} = 0$ is considered first.

If over-regulation can be allowed, it is worthwhile using the flexible Bayes rule with real values for the loss function, i.e. according to equation (11). Such a choice is possible because the determination of the estimator $\hat{m}$ value that is either less than, equal to, or greater than m allows the system state to be brought to the target in a finite time. (However, this time increases approximately proportionally to the difference between the values $\hat{m}$ and m.)

If over-regulation is not allowed, this determination needs to be carried out on the basis of the minimax rule, assuming infinite values of the loss function for $\hat{m} < m$, i.e. using formula (11). This enables over-regulation to be avoided, because it occurs only if $\hat{m} < m$.

Let now $x_{f2} \neq 0$.

The value of the parameter $\hat{m}$ should be determined using the minimax rule with infinite values of the loss function for $\hat{m} < m$, i.e. by dependence (10). Such a choice guarantees that the generation of the inadmissible limit cycles which appear when $\hat{m} < m$ is avoided. If, however, this value is greater than m, the state of the system is brought to the target in a finite time. (Note that in the case $x_{f2} \neq 0$, over-regulation cannot be avoided at all.) A somewhat improved structure can be obtained by dividing the switching region (curve) Q into two parts at the point of its intersection with the axis x_1. For each of them, the values of the parameter $\hat{m}$ should be determined in a different manner. Namely, in the case of the part which lies on the same side of the axis x_1 as the target state, it should be done – as previously – by using the minimax rule with infinite values of the loss function for $\hat{m} < m$, i.e. using formula (10); in the case of the part located on the opposite side, however, by the Bayes rule with real values of the loss function, i.e. according to equation (11). This change does not pose the risk that a cycle will occur, while the use of the flexible Bayes rule makes it possible to render more efficiently the potential sliding process occurring along the part of the switching curve located on the side of the axis x_1 opposite to the target.

If one possesses the value $\hat{m}$ obtained according to the above procedure, the feedback controller equations can be calculated. Thus, the equations of the switching curve K take on the form

$$x_1 = -\frac{\hat{m}}{2}({x_2}^2 - {x_{f2}}^2) + x_{f1} \quad \text{for } x_2 \in (x_{f2}, \infty) \tag{12}$$

$$x_1 = \frac{\hat{m}}{2}({x_2}^2 - {x_{f2}}^2) + x_{f1} \quad \text{for } x_2 \in (-\infty, x_{f2}) \quad . \tag{13}$$

Formula (12) defines the set $K_{-+} = K_{--}$, while dependence (13), the set $K_{+-} = K_{++}$. In the case when, for $x_{f2} \neq 0$, the switching curve is divided into two parts at the point of its intersection with the axis x_1, the equation for the part lying on the side of this axis opposite to the target should be modified as:

$$x_1 = \mathrm{sgn}(x_{f2})(\frac{\hat{m}_b}{2}{x_2}^2 - \frac{\hat{m}}{2}{x_{f2}}^2) + x_{f1} \quad , \tag{14}$$

where $\hat{m}_b$ denotes the additional estimator defining that part, obtained through the Bayes rule with real values of the loss function, i.e. by equation (11). The sets R_- and R_+ constitute adequate areas resulting from the division of the plane R^2 by the curve K, according to formulas (7)-(8). For the sets K_-, K_+, R_-, R_+ obtained in this way, the value of the suboptimal control is defined by the equation

$$u_{\text{hard}}(t) = \begin{cases} -1 & \text{if } [x_1(t), x_2(t)]^{\mathrm{T}} \in (R_- \cup K_-) \\ 0 & \text{if } [x_1(t), x_2(t)]^{\mathrm{T}} \in \{x_f\} \\ +1 & \text{if } [x_1(t), x_2(t)]^{\mathrm{T}} \in (R_+ \cup K_+) \end{cases} \quad , \tag{15}$$

where $[x_1(t), x_2(t)]^{\mathrm{T}}$ means the object state, obtained by a real-time measurement process for any $t \in T$.

3.2 Soft Structure

The control designed in the previous subsection may lead to frequent switchings between the extreme – according to the assumption (A3) – values $+1$ and -1 along sliding trajectories, which should be avoided in mechanical systems, since it can have a negative impact on the endurance of a device and user comfort. Based on the results of Theorem presented in Section 2 and under the condition that the control may take any value in the interval $[-1,1]$, this goal can be obtained by substituting a modified control law, rendered "soft" instead of "hard" (15).

Let the sets K_{--} and K_{+-}, be defined as previously but for the value of the parameter $\hat{m}$ calculated in the previous section for the discontinuous structure. Let also the additional positive constant $\Delta\hat{m}$ be given and the sets K_{-+} and K_{++} be defined for the value $\hat{m} + \Delta\hat{m}$.

As before, the case $x_{f2} = 0$ will be considered first. Let a feedback controller be as follows

$$u_{\text{soft}}(t)=\begin{cases} -1 & \text{if } [x_1(t),x_2(t)]^{\mathrm{T}}\in R_- \\ z(x_1(t),x_2(t)) & \text{if } [x_1(t),x_2(t)]^{\mathrm{T}}\in Q_- \\ 0 & \text{if } [x_1(t),x_2(t)]^{\mathrm{T}}\in \{x_f\} \\ z(x_1(t),x_2(t)) & \text{if } [x_1(t),x_2(t)]^{\mathrm{T}}\in Q_+ \\ +1 & \text{if } [x_1(t),x_2(t)]^{\mathrm{T}}\in R_+ \end{cases} \quad (16)$$

with the function $z:\mathrm{R}^2\to\mathrm{R}$ continuously and strictly increasing from the value -1 on the sets K_{--} and K_{++} to the value +1 on the sets K_{-+} and K_{+-} (see also Fig 1). If the solution $X(\omega,\cdot)$ is "too close" – with respect to real value of the mass – to the set K_{+-}, then control (16) is "too great" and it makes this solution further from the set K_{+-} to the interior of the set Q_+. And inversely, if the solution is "too far" to the set K_{+-}, then control (16) is "too small" and brings the trajectory closer to this set (see also Fig. 1). The result obtained in the above manner is similar to the effect achieved on a bobsled track thanks to the appropriate modeling of its shape. It is a fluid movement, therefore, allowing such a structure to be named "soft". An analogous situation occurs between the sets K_{-+} and K_{--}. The value of the parameter $\Delta\hat{m}$ influences the speed of the control fluctuations in the set Q: the greater the value, the milder the fluctuations. To the primary researches one can suggest $\Delta\hat{m}=\hat{m}/10$.

Having the value $\hat{m}$ obtained according to the material presented in subsection 3.1, and assuming the constant $\Delta\hat{m}$, one can calculate the equation of the set K_{+-}

$$x_1=\frac{\hat{m}}{2}x_2{}^2+x_{f1}-\varepsilon \quad \text{for} \quad x_2\in(-\infty,0) \quad (17)$$

and for the set K_{++}

$$x_1=\frac{\hat{m}+\Delta\hat{m}}{2}x_2{}^2+x_{f1}+\varepsilon \quad \text{for} \quad x_2\in(-\infty,0)\ , \quad (18)$$

where the additional parameter $\varepsilon\geq 0$ is closer to (but is not greater than) precise positioning (i.e. assumed in practice precision of reaching the target state) and has been introduced to avoid the over-increasing of the function z near the axis x_1. The function z can be proposed in the following manner:

$$z(x_1,x_2)=a(x_2)[x_1-c(x_2)]^d-1 \quad \text{for} \quad x_2\in(-\infty,0) \quad (19)$$

with

$$a(x_2)=\frac{4}{\Delta\hat{m}\,x_2{}^2+4\varepsilon} \quad (20)$$

$$c(x_2)=\frac{\hat{m}+\Delta\hat{m}}{2}x_2{}^2+x_{f1}+\varepsilon\ , \quad (21)$$

while the value of the positive parameter d presents a compromise between speed of action of the sub-time-optimal control system and its robustness. Namely, $d=1$ can be treated as neutral; the values $d<1$, results in making the solutions nearer to the curves K_{-+} or K_{++} which slows down the process but increases robustness; and the inverse when $d>1$. For primary experimental research $d=0.25$ is proposed.

The analogous dependencies are outlined in the sets K_{--} and K_{-+}, respectively

$$x_1=-\frac{\hat{m}}{2}x_2{}^2+x_{f1}+\varepsilon \quad \text{for} \quad x_2\in(0,\infty) \quad (22)$$

$$x_1=-\frac{\hat{m}+\Delta\hat{m}}{2}x_2{}^2+x_{f1}-\varepsilon \quad \text{for} \quad x_2\in(0,\infty)\ . \quad (23)$$

The function z can be proposed here as

$$z(x_1,x_2)=a(x_2)[x_1-c(x_2)]^{1/d}-1 \quad \text{for} \quad x_2\in(0,\infty) \quad (24)$$

with

$$a(x_2)=\frac{-4}{\Delta\hat{m}\,x_2{}^2+4\varepsilon} \quad (25)$$

$$c(x_2)=-\frac{\hat{m}}{2}x_2{}^2+x_{f1}+\varepsilon\ . \quad (26)$$

Let now $x_{f2}\neq 0$. The concept introduced in the preceding paragraph should be transferred here in a natural way. For simplicity of notation, the case $x_{f2}>0$ will be investigated below; if $x_{f2}<0$ considerations are symmetrical. A feedback controller is also defined here by formula (16).

The sets K_{+-} and K_{++} in the part between the target and the axis x_1, should be given as for the hard structure, both defined by the equation

$$x_1=\frac{\hat{m}+\Delta\hat{m}}{2}(x_2{}^2-x_{f2}{}^2)+x_{f1} \quad \text{for} \quad x_2\in[0,x_{f2}) \quad (27)$$

with

$$z(x_1,x_2)=1 \quad \text{for} \quad x_2\in[0,x_{f2})\ . \quad (28)$$

For the part lying in the lower half-plane, the set K_{++} is defined by

$$x_1=\frac{\hat{m}+\Delta\hat{m}}{2}(x_2{}^2-x_{f2}{}^2)+x_{f1} \quad \text{for} \quad x_2\in(-\infty,0) \quad (29)$$

and the set K_{+-} by

$$x_1=\frac{\hat{m}}{2}x_2{}^2-\frac{\hat{m}+\Delta\hat{m}}{2}x_{f2}{}^2+x_{f1}-\varepsilon \quad \text{for} \quad x_2\in(-\infty,0)\ . \quad (30)$$

The function z is given as

$$z(x_1, x_2) = a(x_2)[x_1 - c(x_2)]^d - 1 \quad \text{for} \quad x_2 \in (-\infty, 0) \tag{31}$$

with

$$a(x_2) = \frac{-4}{\Delta\hat{m}\, x_2^{\,2} + 4\varepsilon} \tag{32}$$

$$c(x_2) = \frac{\hat{m} + \Delta\hat{m}}{2}(x_2^{\,2} - x_{f2}^{\,2}) + x_{f1} + \varepsilon \quad . \tag{33}$$

Finally, the sets K_{--} and K_{-+} are defined by

$$x_1 = -\frac{\hat{m}}{2}(x_2^{\,2} - x_{f2}^{\,2}) + x_{f1} + \varepsilon \quad \text{for} \quad x_2 \in (x_{f2}, \infty) \tag{34}$$

$$x_1 = -\frac{\hat{m} + \Delta\hat{m}}{2}(x_2^{\,2} - x_{f2}^{\,2}) + x_{f1} - \varepsilon \quad \text{for} \quad x_2 \in (x_{f2}, \infty) \quad , \tag{35}$$

respectively, and the function z is given as

$$z(x_1, x_2) = a(x_2)[x_1 - c(x_2)]^{1/d} - 1 \quad \text{for} \quad x_2 \in (x_{f2}, \infty) \tag{36}$$

with

$$a(x_2) = \frac{-4}{\Delta\hat{m}(x_2^{\,2} - x_{f2}^{\,2}) + 4\varepsilon} \tag{37}$$

$$c(x_2) = -\frac{\hat{m}}{2}(x_2^{\,2} - x_{f2}^{\,2}) + x_{f1} + \varepsilon \quad . \tag{38}$$

Finally, frequent switchings of the control along sliding trajectories were eliminated, according to the assumed goal of the soft structure. The control changes its value fluently in full range of the interval $[-1,1]$.

4. FINAL SUGGESTIONS

The theorem presented in Section 2 was formulated in its basic form. The concept of resulting suboptimal structures can be easily supplemented and transposed for a number of new aspects that recur in engineering practice.

The heretofore basic form of the controlling structures may lead to unacceptably high velocities, if the distance between the initial and target points is too long. An analysis of the problem considered here undergoes little modification if one takes into account a velocity limitation to the value $z > 0$, i.e.

$$|\dot{s}(t) \leq z| \quad . \tag{39}$$

As a next example consider the task of modeling the dynamic of a drive. Let u introduced in equation (1) mean then the moment obtained from the drive, which is treated as an inertial element with the constant $\mathcal{T} > 0$, i.e.

$$\dot{u}(t) = -\frac{1}{\mathcal{T}} u(t) + v(t) \tag{40}$$

and subsequently v be a bounded control. If the number $\mathcal{T}$ is considered as a stochastic process, the concept presented in this paper can easily be transposed to a system formed thus.

Analysis remains analogous if equation (1) is supplemented with the discontinuous model of motion resistance $-w\,\mathrm{sgn}(\dot{s}(t))$, therefore when it takes the form

$$m\ddot{s}(t) = u(t) - w\,\mathrm{sgn}(\dot{s}(t)) \quad , \tag{41}$$

where $w \in [0,1)$ and one treats it as a stochastic process.

Note that the parameters $\mathcal{T}$ and especially w constitute the reflection of an entire array of physical phenomena, reduced to a single constant due to the necessity to simplify the model. Then the issue consists here not – as in the case of the load m – in approaching the unknown real value (since no such thing exists), but in specifying the best possible characterisation of these phenomena by a random concept.

For further details see the article (Kulczycki et al, 2004), where one can also find, among others, results of experimental verification confirming the correctness of the method presented in this paper.

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