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COMMUNICATIONS

FINITE NUMERICAL SEQUENCES AS A MATHEMATICAL MODEL OF DYNAMIC STATES IN ELECTRICAL RLC CIRCUITS

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The paper deals with methods of finding a solution in transient states. The method, which is the main subject of our investigations, describes an electrical circuit using the finite numerical sequences $[q_0, q_1, ..., q_{2n-1}]$, representing the charge in this circuit with strictly defined rules and properties. The values of this sequences are independent of time and strongly connected with a circuit topology.

K e y w o r d s: Structure theory, Special Sequences and Polynomials, Solutions of Linear Differential Equations

1 INTRODUCTION

The paper deals with three methods of finding a solution in transient states. The first operator method is based on the Laplace transform. The solution of the model is the transfer function $T(s)$. The second method is based on differential equations, as a result time functions describing the current, are presented. The third method, which is the main subject of investigations, describes an electrical circuit using the finite numerical sequences $[q_0, q_1, ..., q_{2n-1}]$, representing the charge in this circuit with strictly defined rules and properties. The values of this sequences are independent of time and strongly connected with a circuit topology. The solving algorithm can be translated into any programming language or into simple microchips.

2 METHOD USING THE LAPLACE TRANSFORM

In this method, the electrical circuit consisting only of resistors, coils, capacitors and of course a power supply is presented as a "black box" with an input and output. Some input signals are attached, and the most important information is what response of this circuit can be obtained on the output. The relationship, the so-called transfer function between the input and output signals is given in the Laplace domain as a fraction with the output signal as a numerator and the input signal as a denominator. The input signal is attached, all of the properties of the circuit are hidden and represented by the transfer function. As the output, the signal in the Laplace domain is given.

Let the transfer function be defined by the formula (1). $Y_{n-1}(s)$ is the $(n-1)$-order polynomial of the output signal in the Laplace domain, and $U_n(s)$ is the n-order polynomial of the input signal in the Laplace domain. The number n is the order of the model.

$$
\begin{aligned}
T_n(s) = \frac{Y_{n-1}(s)}{U_n(s)} &= \frac{\sum_{i=0}^{n-1} b_{ni} s^i}{\sum_{i=0}^{n} a_{ni} s^i} \\
&= \frac{b_{nn-1}s^{n-1} + b_{nn-2}s^{n-2} + ... + b_{n1}s + b_{n0}}{a_{nn}s^n + a_{nn-1}s^{n-1} + ... + a_{n1}s + a_{n0}} \qquad (1)
\end{aligned}
$$

where: $a_{ni}, b_{ni} \in \Re^+$

The transfer function allows to find the response of the circuit. In our investigations, typical elements with their own, strictly defined transfer functions are used. When the electrical circuit contains many elements, finding the main transfer function is necessary. It can be found by grouping all of transfer functions with well defined rules and properties. Very often in the investigations only two of elementary properties are used. When the main transfer function is found, the output can be obtained in a simple way — the transfer function should be multiplied by the input signal. Of course, the result is given in the Laplace domain. In order to get the result in the time domain, inverse Laplace transform is used.

3 METHOD USING TIME FUNCTIONS

This method is in a sense equivalent to the method which uses Laplace transformation. For frequency changing electrical circuits there is no definition of a transfer function in the time domain. In this method, linear electrical circuits only with resistors, coils, capacitors and

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power supplies are considered. There are the input signals as time functions, so differential and integral equations are needed. Because of using differential and integral equations, the solutions are sometimes hard to find, and from this point this method of consideration the electrical circuits is used in more simple cases.

4 METHOD USING FINITE NUMERICAL SEQUENCES

Finite Numerical Sequences are the main subject of this paper. By this method, in order to analyze electrical circuits, special transforms are presented. The method of analyzing contains only four mathematical operations: adding, subtraction, multiplication and division. Let $f_n(t)$ denote the original of a transfer function. It can be presented as Maclaurin's sequence:

$$f_n(t) = \sum_{k=0}^{\infty} c_{nk} t^k = \sum_{k=0}^{\infty} \frac{1}{k!} A_{nk} t^n \tag{2}$$

The coefficients A_{nk} are the functions of all polynomials coefficients, which are the transfer function (formula (3)).

$$A_{nk}(a_{n0}, a_{n1}, ..., a_{nn-1}, b_{n0}, b_{n1}, ..., b) \tag{3}$$

On the other hand, the coefficients a_{ni} and b_{ni} are the functions of all coefficients A_{nk}:

$$\begin{aligned} a_{ni} &= a_{ni}(A_{n0}, A_{n1}, \dots, A_{nk}, \dots) \\ b_{ni} &= b_{ni}(A_{n0}, A_{n1}, \dots, A_{nk}, \dots) \end{aligned} \tag{4}$$

There is a conclusion that there are some relationships between A_{nk} and a_{ni} and b_{ni}. These relationships allow to find the way from the original of the transfer function to the transfer function and inversely. First let us consider transformation from the time function to the transfer function. According to the theory of symmetrical polynomials [1] it can be proven that relationships (4) for the few first orders of the model are:
for $n = 1$

$$\begin{pmatrix} b_{10} \\ b_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -A_{10} \end{pmatrix}^{-1} \cdot \begin{pmatrix} A_{10} \\ A_{11} \end{pmatrix}, \tag{5}$$

for $n = 2$

$$\begin{pmatrix} b_{21} \\ b_{20} \\ a_{21} \\ a_{20} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -A_{20} & 0 \\ 0 & 0 & -A_{21} & -A_{20} \\ 0 & 0 & -A_{22} & -A_{21} \end{pmatrix}^{-1} \cdot \begin{pmatrix} A_{20} \\ A_{21} \\ A_{22} \\ A_{23} \end{pmatrix}, \tag{6}$$

for $n = 3$

$$\begin{pmatrix} b_{32} \\ b_{31} \\ b_{30} \\ a_{32} \\ a_{31} \\ a_{30} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -A_{30} & 0 & 0 \\ 0 & 0 & 1 & -A_{31} & -A_{30} & 0 \\ 0 & 0 & 0 & -A_{32} & -A_{31} & -A_{30} \\ 0 & 0 & 0 & -A_{33} & -A_{32} & -A_{31} \\ 0 & 0 & 0 & -A_{34} & -A_{33} & -A_{32} \end{pmatrix}^{-1} \begin{pmatrix} A_{30} \\ A_{31} \\ A_{32} \\ A_{33} \\ A_{34} \\ A_{35} \end{pmatrix} \tag{7}$$

etc.

In that transformation, there is a very important assumption that the determinant of the so-called decision matrix is not equal to zero. The presented transformation gives the possibility to look at the circuit from the point of view of synthesis. There are some problems in which only the time response function is given and we are looking for parameters describing the circuit. One of this is the transfer function. The presented transformation is very useful because the behaviour of the simplified model is as exact as the real model. This method is also useful in the investigations of the transient states, where the most important are the first time periods of response of the circuit. In practice, all numerical calculations are connected with some errors. The source of these errors is that we lose some part of information by calculating in a finite range whereas the mathematical model consists of an infinite number of desired elements or calculations.

Now let us think about the inverse transformation: from the transfer function to its original in the time domain. This inverse transformation is quite equivalent to the first one. It can be also proven, according to the theory of symmetrical polynomials, that relationships (3) for the few first orders of a model are:
for $n = 1$

$$\begin{pmatrix} A_{10} \\ A_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_{10} & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} b_{10} \\ 0 \end{pmatrix} \tag{8}$$

for $n = 2$

$$\begin{pmatrix} A_{20} \\ A_{21} \\ A_{22} \\ A_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 \\ a_{20} & a_{21} & 1 & 0 \\ a_{20} & a_{21} & 1 & \end{pmatrix}^{-1} \cdot \begin{pmatrix} b_{21} \\ b_{20} \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

for $n = 3$

$$\begin{pmatrix} A_{30} \\ A_{31} \\ A_{32} \\ A_{33} \\ A_{34} \\ A_{35} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a_{32} & 1 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 1 & 0 & 0 & 0 \\ a_{30} & a_{31} & a_{32} & 1 & 0 & 0 \\ 0 & a_{30} & a_{31} & a_{32} & 1 & 0 \\ 0 & 0 & a_{30} & a_{31} & a_{32} & 1 \end{pmatrix}^{-1} \begin{pmatrix} b_{32} \\ b_{31} \\ b_{30} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

etc.

This transformation allows to look at the electrical circuit in the time domain. It is a "user friendly" domain because it can easily visualize the considered problem.

The main advantages of both presented transformations are:

– the fact that all calculations are performed with finite numerical sequences and, which is the most important, without losing any information,
– the presented transformations are very accurate,
– the way to get the results is fast and simple.

5 ALGEBRA OF FINITE SEQUENCES

Let us have: X, Y finite sequences, $q_0, \dots, q_s$, $w_0, \dots, w_r$, items of decision matrix D (7). Let a be a real number. The number of items in each sequence must be always even.

Definition of zero-item (null sequence):

$$0 \quad \text{or} \quad [\ \ ]. \tag{11}$$

Definition of multiplication of a finite sequence by a real number:

$$a \cdot [q_0, q_1, \dots, q_s] = [a \cdot q_0, a \cdot q_1, \dots, a \cdot q_s] \tag{12}$$

Definition of finite sequence addition:
addition consists of three calculations:

– Expanding each sequence to $r+s$ items,

$$\begin{aligned} X_{expand} &= [q_0, q_1, \dots, q_s, q_{s+1}, \dots, q_{s+r}] \\ Y_{expand} &= [w_0, w_1, \dots, w_s, w_{s+1}, \dots, w_{s+r}] \end{aligned} \tag{13}$$

– Elementwise addition — as a result a new $s+r$ elements sequence is obtained:

$$X_{expand} + Y_{expand} = [q_0 + w_0, q_1 + w_1, \dots, q_{r+s} + w_{r+s}] \tag{14}$$

– Contraction. New shorter sequence $X_{expand} + Y_{expand}$ is needed.

$$X + Y = [c_0, c_1, \dots, c_{2k-1}] \qquad 2k-1 < r+s. \tag{15}$$

The contraction and expanding operations are describing in [1].

Definition of derivative
The first order derivative of X is denoted by X'. If $X = [q_0, q_1, \dots, q_{2s-1}]$, then its derivative is

$$[q_0, q_1, ..., q_{2s-1}]' = [q_1, ..., q_{2s-2}]. \tag{16}$$

Derivatives of higher orders are based on the definition of the first derivative of a finite sequence:

$$X^{(n)} = \left(X^{(n-1)}\right)' \qquad n = 1, 2, 3, \dots, \ X(0) = X \tag{17}$$

6 CONNECTION OF FINITE SEQUENCES WITH TIME FUNCTION

For time function, the derivative in point zero is very well known because it is an algorithmic operation. Items $f(0), f'(0), f''(0), \dots, f^{(2n-1)}(0)$ are elements of finite sequence. The most important question is how many items are needed. To answer this question, a decision matrix is necessary. The first element which makes the determinant of the decision matrix equal to zero, is left.

Examples of time functions:

$$\begin{aligned} f(t) = 0 &\longleftrightarrow [\ ] \\ f(t) = 1 &\longleftrightarrow [1,0] \\ f(t) = t &\longleftrightarrow [0,1,0,0] \\ f(t) = t^2 &\longleftrightarrow [0,0,2,0,0,0] \\ f(t) = \sin(t) &\longleftrightarrow [0,1,0,\text{-}1] \\ f(t) = e^{-t} &\longleftrightarrow [1,\text{-}1] \\ f(t) = e^{-2t} &\longleftrightarrow [1,\text{-}2] \\ f(t) = e^{-t}\sin(2t) &\longleftrightarrow [1,1,1,\text{-}2,1,0,1,\text{-}2,1,0] \end{aligned} \tag{18}$$

7 EXAMPLE

In this serial circuit, the resistor ($R = 5\,\Omega$), capacitor ($C = 1/6$ F) and coil ($L = 1$ H) are given. The capacitor C and the coil L have their initial conditions: the current I in the coil and the charge q on the capacitor. Knowing that the charge is the first derivative of the current, a change of the initial condition to a charge function is needed. The equation of the circuit is given:

$$\begin{gathered} Lq'' + Rq' + \frac{1}{C}q = 0 \\ q(0) = 2 \quad \text{and} \quad q'(0) = 3 \end{gathered} \tag{19}$$

In this example initial conditions have a special function. After switching on a jumper there are a source of power and input signal. The current in the circuit as a response is needed. The following formula shows how many elements in sequence are wanted. K =number items = $2*$(order diff. eq. + order of supply). In this example $K = 2*(2+0) = 4$ items. The first and second elements of the sequence are known from the initial conditions $q(0) = 2$, $q'(0) = 3$, next elements are computed from differential equation (19): $q''(0) = -5q'(0) - 6q(0) = -27$. The fourth and fifth element are $q'''(0) = [q''(0)]' = -5q'' - 6q' = 117$ and $q^{IV}(0) = [q'''(0)]' = -5q''' - 6q'' = 423$. In this way the full sequence is given: $[2, 3, -27, 117]$.

7.1 Transfer Function

By using formula (7), the decision matrix is made, but to check the order of the model higher determinant unequal to zero must be made. Decision matrix [2]

$$D_1 = \begin{pmatrix} 1 & 0 \\ 0 & -q_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},$$

$\det D_1 = -2$,

$$D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -q_0 & 0 \\ 0 & 0 & -q_1 & -q_0 \\ 0 & 0 & -q_2 & -q_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & -27 & -3 \end{pmatrix},$$

$\det D_2 = -63\,,$

$$D_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -q_0 & 0 & 0 \\ 0 & 0 & 1 & -q_1 & -q_0 & 0 \\ 0 & 0 & 0 & -q_2 & -q_1 & -q_0 \\ 0 & 0 & 0 & -q_3 & -q_2 & -q_1 \\ 0 & 0 & 0 & -q_4 & -q_3 & -q_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 27 & -3 & -2 \\ 0 & 0 & 0 & -117 & 27 & -3 \\ 0 & 0 & 0 & -423 & -117 & 27 \end{pmatrix},$$

$$\det D_3 = 0\,, \tag{21}$$

From formula (10), by using the inverse of the decision matrix the coefficients of the transfer function are given.

$$\begin{pmatrix} b_1 \\ b_0 \\ a_1 \\ a_0 \end{pmatrix} = D_2^{-1} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & -27 & -3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 3 \\ -27 \\ 117 \end{pmatrix} = \begin{pmatrix} 2 \\ 13 \\ 5 \\ 6 \end{pmatrix} \tag{22}$$

The transfer function is based on coefficients from (22)

$$Q(s) = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} = \frac{2s + 13}{s^2 + 5s + 6}. \tag{23}$$

7.2 Time Function

Each time function can be presented as a sum of infinite elements which are decreasing like Maclaurin's sequence.

$$q(t) = q_0 + q_1 t + \frac{q_2}{2!}t^2 + \frac{q_3}{3!}t^3 + \dots$$
$$q(t) = q_0 + q_1 t + \frac{q_2}{2!}t^2 + \frac{q_3}{3!}t^3 = 2 + 3t - \frac{27}{2}t^2 + \frac{117}{3!}t^3\,. \tag{24}$$

From classical calculation based on inverse Laplace transform (L^{-1}) the time function is given:

$$L^{-1}[Q(s)] = q(t) = 9e^{-2t} - 7e^{-3t}\,. \tag{25}$$

Using the definition of Maclaurin's sequence for $\exp(t)$ each element of (25) can be computed.

$$\begin{aligned} 9e^{-2t} &= 9\left(1 - 2t + \tfrac{4t^2}{2} - \tfrac{8t^3}{6}\right) \\ -7e^{-3t} &= -7\left(1 - 3t + \tfrac{9t^2}{2} - \tfrac{27t^3}{6}\right) \end{aligned} \tag{26}$$

As a result of the last operation, the "finite" time function like (24) is given. In this part of calculation some kind of approximation is presented but on the other hand all information about the circuit is saved. Coefficients from the time function are necessary to make a finite sequence [3] or a transfer function describing our circuit without any approximation. The coefficients $q_4, q_5 \dots$ are linear combination of q_0 to q_3, this observation is called a fractal effect [1].

8 CONCLUSIONS

The finite numerical sequences can be used to solve linear differential equations. This method is very useful as a substitute for finding Laplace Transform. In classical way integration and differential operations are needed. Solving some of them in analytic calculations are not possible. In this model only few operations on matrix are uses: adding, subtracting, inverting, checking determinant and multiplying by a number or by a vector. This method can test the initial phase in the transient states. The transient states are interesting in the first phases after switching on or switching off the current in a circuit. These algorithms can be implemented in assembler programs, which can check parameters of a circuit on-line.

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