ON THE SYNTHESIS OF TIME-OPTIMAL POSITIONAL CONTROLLER

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ABSTRACT

This paper concerns the problem of time-optimal positional control of discontinuous dynamic objects. The random concept of solving this task is presented.

Namely, the problem of time-optimal control of the system, whose dynamics are described by the random differential equation:

$$\ddot{Y}(\omega,t) = V(\omega,t)\cdot f(\dot{Y}(\omega,t)) + U(\omega,t) \ ,$$

where U represents a control, V is a given stochastic process with almost all realizations being continuous and jointly bounded, f denotes a real piecewise continuous function, is considered. The expression $V(\omega,t)\cdot f(\dot{Y}(\omega,t))$ represents a model of motion resistances.

As a practical conclusion, a suboptimal control structure defined as a closed-loop system with a deterministic feedback controller function, is proposed. Its construction is based on statistical decision theory. The paper also contains suggestions for practical applications.

INTRODUCTION

There is a broad class of industrial devices which realize their technological cycles mainly through a change of the position in particular mechanisms, e.g. tracer machines, automata and robots, saddles of machine tools and many others. The control yielding minimum operation time directly influences the efficiency of such devices. Other examples are various security and failure devices; the shortest possible time of reaction is a basic element of their reliability. The above classes of objects and controllers are called positional.

The dynamics of those objects are described by the following differential equation:

$$\ddot{y}(t) = h(\dot{y}(t),y(t),t) + u(t) \ , \qquad (1)$$

where u is a bounded control function, and y denotes the position of that object. The essential element of the above model is the bounded function h, which represents motion resistances. For the majority of practically appearing types of motion resistances, that function can be expressed in the form:

$$h(\dot{y}(t),y(t),t) = v(\dot{y}(t),y(t),t)\ f(\dot{y}(t)) \ , \qquad (2)$$

where v denotes a continuous function, and f is a piecewise continuous function. The discontinuity introduced by the function f makes it impossible to apply the methods of the classical theory of dynamic optimization [1]. Naturally, if the forms of the functions f and especially v are more complex, then the synthesis of the time-optimal control and the analysis of the system as well as the consecutive identification of parameters which define the above functions, become disproportionally more difficult. Practically, besides the trivial forms of the function v (e.g. a constant function in [2]), the synthesis of the time-optimal control system turns out to be impossible to realize in the case of such a deterministic task.

In this paper a probabilistic concept of solution will be proposed. In the model of motion resistances adopted here, it is assumed that the function v defined in equation (2) is the realization of some given stochastic process with almost all realizations being continuous and jointly bounded. Therefore, the dependence of the function v on $\dot{y}(t)$, $y(t)$ and t, is substituted by the dependence on a random factor. Moreover, such a model of motion resistances also regards, in the form of probabilistic undeterminateness, the dependence of motion resistances on a number of other factors, not only $\dot{y}(t)$, $y(t)$ and t, but also those which are usually omitted in the deterministic approach due to the necessity to simplify the model. The probabilistic concept also considers the perturbations occurring in the system.

THEOREM

Some notions used in the following, will be precised now.

Let T be an interval with nonempty interior, and let $t_0 \in T$.

First, the deterministic differential equation:

$$\dot{x}(t) = g(x(t),t) \quad (3)$$

with the initial condition:

$$x(t_0) = x_0 , \quad (4)$$

where $g : \mathbb{R}^n \times T \dashrightarrow \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$, will be considered. Solutions of differential equations in three different senses, which are usually used in the analysis of discontinuous dynamical systems, will be presented now.

The function $x : T \dashrightarrow \mathbb{R}^n$, absolutely continuous on every compact subinterval of the set T, is a solution of differential equation (3):

- in the Caratheodory sense (C-solution), if it fulfills equation (3) almost everywhere in T,
- in the Filippov sense (F-solution), if:

$$\dot{x}(t) \in F[g](x(t),t) \quad \text{almost everywhere in } T , \quad (5)$$

- in the Krasovski sense (K-solution), if:

$$\dot{x}(t) \in K[g](x(t),t) \quad \text{almost everywhere in } T , \quad (6)$$

where the operators F and K are defined in the following way:

$$F[g](x(t),t) = \bigcap_{e>0} \bigcap_{\substack{Z\subset\mathbb{R}^n \\ m(Z)=0}} \operatorname{conv}[g((x(t)+eB)\setminus Z,t)] \quad (7)$$

$$K[g](x(t),t) = \bigcap_{e>0} \operatorname{conv}[g(x(t)+eB,t)] \ ; \quad (8)$$

B denotes the open unit ball in the space $\mathbb{R}^n$, m is the Lebesgue measure, and conv[A] means the convex closed hull of the set A.

The C-, F- or K-solutions of deterministic differential equation (3) with initial condition (4) are unique, if all C-, F- or K-solutions, respectively, are identically equal functions.

Any C-solution is a K-solution and any F-solution is a K-solution, but no other relations occur among the C- and F-solutions, however K-solutions are a very large class. The result is that in the consideration of differential equations with a discontinuous right-hand side, a considerable difficulty is presented by the lack of a universal concept of a solution of that type of equations. Of course, the existence of unique and equal to each other C-, F- and K-solutions, greatly simplifies further analysis. That fact is especially worthwhile noticing.

The above concepts of solutions will be generalized to random differential equations, in the following. Such a generalization, however, is not unique. In this paper the concept of almost certain (with probability 1, first type) solutions will be applied, because of its trivial interpretation.

Let (Ω,Σ,P) be a probability space.

The following random differential equation:

$$\dot{X}(\omega,t) = G(\omega,X(\omega,t),t) , \quad (9)$$

with the initial condition:

$$X(\omega,t_0) = X_0(\omega) \quad \text{for almost all } \omega \in \Omega , \quad (10)$$

where $G : \Omega \times \mathbb{R}^n \times T \dashrightarrow \mathbb{R}^n$ and X_0 is an n-dimensional random variable defined on (Ω,Σ,P), will be considered now.

An n-dimensional stochastic process X defined on (Ω,Σ,P) and T, is an almost certain C-, F- or K-solution of random differential equation (9), if almost all its realizations are C-, F- or K-solutions, respectively, of the related deterministic differential equations received at the fixed $\omega \in \Omega$.

Almost certain C-, F- or K-solutions of random differential equation (9) with initial condition (10) are unique, if all almost certain C-, F- or K-solutions, respectively, are equivalent stochastic processes (or stochastic modifications).

The generalization of the concept of time-optimal control to random systems is not unique, either. From a practical view point, it would be the most useful to define a control being only a function of time and the state in closed-loop systems, realizing the minimum of the expected value of the time of reaching the target set. Unfortunately, the task so formulated does not provide hope for its solution.

In the following, a different definition of the time-optimal control for random systems, will be formulated. That control, by analogy to the almost certain solution, will be called an almost certain time-optimal control.

Let $U_a \subset \{u : T \dashrightarrow \mathbb{R}^m\}$ represent a set of admissible controls. Let also the function $G : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times T \dashrightarrow \mathbb{R}^n$ which define the dynamics of the random system submitted to the control U:

$$\dot{X}(\omega,t) = G(\omega,X(\omega,t),U(\omega,t),t) \quad (11)$$

with initial condition (10), be given. Then, the m-dimensional stochastic process U^o defined on (Ω,Σ,P) and T, will be called an almost certain time-optimal control, if almost all its realizations are time-optimal controls for proper deterministic systems obtained at the fixed $\omega \in \Omega$.

If the target set is the origin of coordinates, a time-optimal control is called a time-optimal stabilizing control.

An almost certain time-optimal control ensures realization of the minimum of the expected value of the time of reaching a target set, however, it is additionally dependent on the random factor. The result of this dependent is that the above control is difficult to apply directly, but it constitutes a useful basis for the creation of technical constructions of suboptimal structures in which the direct dependency of the control function on the random factor is eliminated.

In [3] the following theorem, which thesis is a solution of the random concept of solving the problem of time-optimal positional control, has been proved.

THEOREM

Let $t_0 \in \mathbb{R}$, $T = [t_0, \infty)$, $x_0 \in \mathbb{R}^2$. Let (Ω, Σ, P) be a complete probability space, $v_-, v_+ \in \mathbb{R}$ be such that $-1 < v_- \le v_+ < 1$, and $U_a = \{u : T \dashrightarrow [-1,1]\}$ represent a set of admissible controls. Let also V be a real stochastic process defined on (Ω, Σ, P) and T, with almost all realizations being continuous and fulfilling boundary condition $V(\omega,t) \in [v_-, v_+]$ for $t \in T$. Let $f : \mathbb{R} \dashrightarrow [-1,1]$ be a piecewise continuous function, locally Lipschitz except at discontinuous points, and such that $z \cdot f(z) \ge 0$ for every $z \in \mathbb{R}$. Finally, let be given the random differential equation describing the dynamics of the system submitted to the control U:

$$\dot{X}_1(\omega,t) = X_2(\omega,t) \tag{12}$$

$$\dot{X}_2(\omega,t) = U(\omega,t) - V(\omega,t)\, f(X_2(\omega,t)) \tag{13}$$

and the initial condition:

$$\begin{bmatrix} X_1(\omega,t_0) \\ X_2(\omega,t_0) \end{bmatrix} = X_0(\omega) \text{ for almost all } \omega \in \Omega\,, \tag{14}$$

where X_0 denotes a two-dimensional random variable defined on (Ω, Σ, P), with a one-point distribution concentrated at x_0.

Then, there exists an almost certain time-optimal stabilizing control U^o, which realizations take on the values +1, -1 and have at most one discontinuity point; it generates a unique almost certain C-solution, which is also a unique almost certain F-solution and a unique almost certain K-solution. ■

In the proof of the above theorem [3] the state space has been subdivided into following disjoint sets: $\{[0,0]^T\}$, Q_+, Q_-, R_+ and R_- (Fig. 1). First, let K_{+-}, K_{++}, denote sets of all states which can be brought to the origin by the control $u \equiv +1$, if $v \equiv v_-$ or $v \equiv v_+$, respectively; analogically, K_{--} and K_{-+} for $u \equiv -1$, if $v \equiv v_-$ or $v \equiv v_+$, respectively.

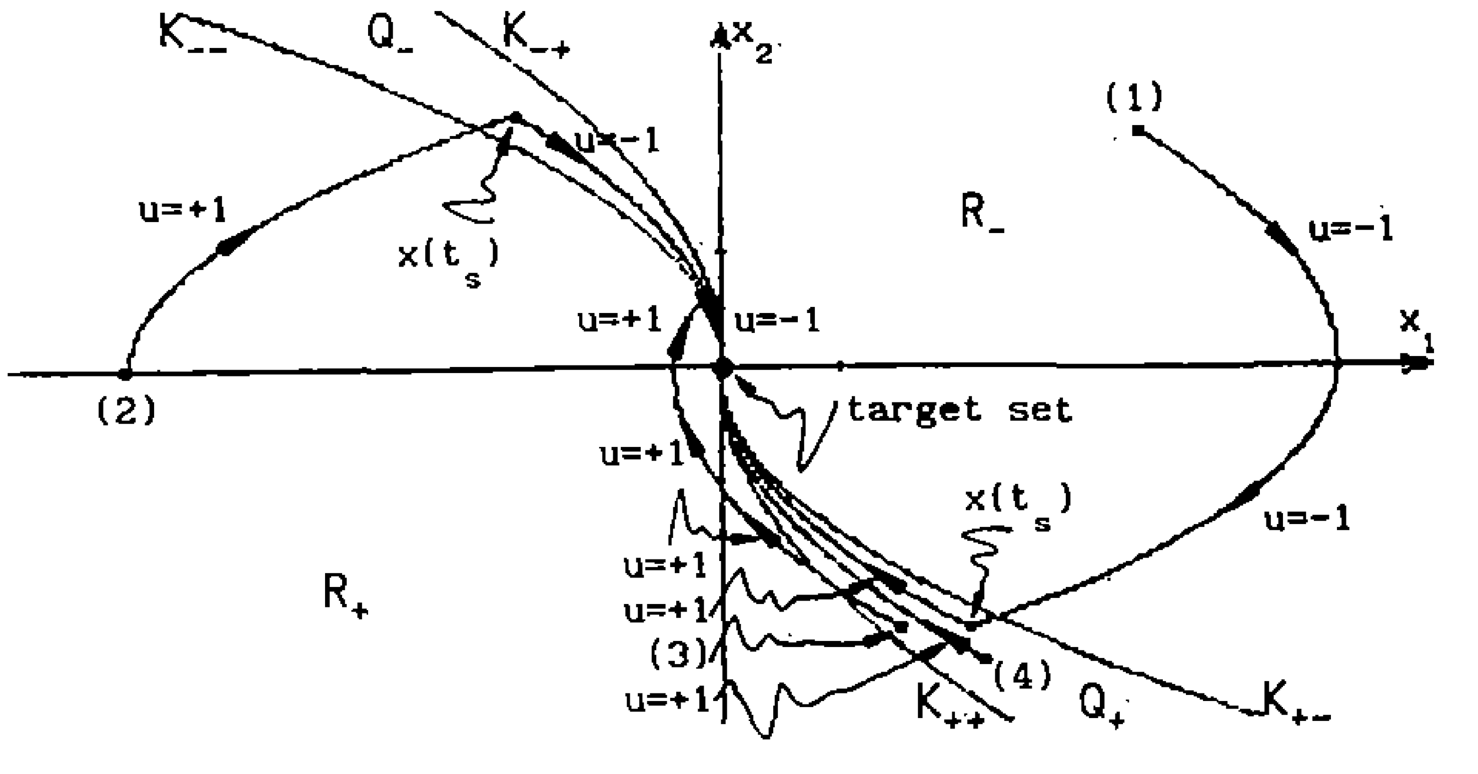

Fig. 1

Next, let the following sets be given:

$$Q_+ = \{ [x_1,x_2]^T \in \mathbb{R}^2 \text{ such that there exist } [x_1',x_2]^T \in K_{++} \text{ and } [x_1'',x_2]^T \in K_{+-} \text{ with } x_1' \le x_1 \le x_1'' \} \tag{15}$$

$$Q_- = \{ [x_1,x_2]^T \in \mathbb{R}^2 \text{ such that there exist } [x_1',x_2]^T \in K_{--} \text{ and } [x_1'',x_2]^T \in K_{-+} \text{ with } x_1' \le x_1 \le x_1'' \} \tag{16}$$

$$R_+ = \{ [x_1,x_2]^T \in \mathbb{R}^2 \backslash Q \text{ such that there exists } [x_1',x_2]^T \in Q \text{ with } x_1 < x_1' \} \tag{17}$$

$$R_- = \{ [x_1,x_2]^T \in \mathbb{R}^2 \backslash Q \text{ such that there exists } [x_1',x_2]^T \in Q \text{ with } x_1' < x_1 \}\,, \tag{18}$$

where $Q = Q_+ \cup \{[0,0]^T\} \cup Q_-$.

Let now $\omega \in \Omega$, or also $V(\omega,\cdot)$, be fixed. If $x_0 \in R_-$, there exists t_s such that the solution generated by the control:

$$u^o(t) = \begin{cases} -1 & \text{for } t \in [t_0, t_s) \\ +1 & \text{for } t \in [t_s, \infty) \end{cases} \tag{19}$$

reaches the origin in minimum time t_f, with $t_0 < t_s < t_f < \infty$ and $x(t_s) \in Q_+$ (Fig. 1; the trajectory marked (1)). Analogically, if $x_0 \in R_+$, there exists t_s such that the solution generated by the control:

$$u^o(t) = \begin{cases} +1 & \text{for } t \in [t_0, t_s) \\ -1 & \text{for } t \in [t_s, \infty) \end{cases} \qquad (20)$$

reaches the origin in minimum time t_f, with $t_0 < t_s < t_f < \infty$ and $x(t_s) \in Q_-$ (Fig. 1; the trajectory marked (2)). Let now $x_0 \in Q_+$; for particular $\omega \in \Omega$ the time-optimal control can be of the form (19) with $x(t_s) \in Q_+$ (Fig. 1; analogical to the trajectory marked (1)) or (20) with $x(t_s) \in Q_-$ (Fig. 1; the trajectory marked (3)) or the following:

$$u^o(t) = +1 \quad \text{for } t \in [t_0, \infty) \qquad (21)$$

(Fig. 1; the trajectory marked (4)). The case $x_0 \in Q_-$ is similar; the counterpart of control (21) is:

$$u^o(t) = -1 \quad \text{for } t \in [t_0, \infty) \; . \qquad (22)$$

The proof of time-optimality of the above described functions u^o is based on the theory of differential inequalities [3].

Finally, the function $U^o : \Omega \times T \dashrightarrow \{-1,+1\}$ defined by the formula $U^o(\omega,\cdot) \equiv u^o$, where u^o was assigned above at fixed $\omega \in \Omega$, is a stochastic process, and the wanted almost certain time-optimal stabilizing control fulfilling the conditions formulated in the thesis of this theorem [3].

The changes of the sign of particular realizations of the control function U^o (switches of the control) occur only when the state of the system is placed in the set (closed area) $Q = Q_- \cup \{[0,0]^T\} \cup Q_+$ (Fig. 1). That is why the above set is called the switching area.

REMARKS

The switching curve γ, well known from the classical case of the transfer of a mass [1, Chapter 7.2], was generalized above to the closed area Q. It is worthwhile noticing, that $\gamma \subset Q$ only if $0 \in [v_-, v_+]$ which is never true in practice. Of course, the additional condition $v_- = v_+ = 0$ implies that $Q = \gamma$.

As it was mentioned, besides specific cases, a direct construction of the system generating an almost certain time-optimal stabilizing control encounters difficulties. The value of that control is in fact directly dependent on the random factor, which is unknown *a priori*. However, thanks to the facts proved in the above quoted theorem, the presented material is a suitable basis for the creation of technical solutions of suboptimal structures, in which such a dependence is eliminated.

To give an example: the function U^o is a stochastic process with bounded values, therefore, there exists its expected value which, being a deterministic function, can be used in the process of a suboptimal control.

Another simple example: in the proof of the theorem presented in this paper, it was also shown [3] that for a fixed $x_0 \in (R_- \cup R_+)$ the time of change of the sign in particular realizations of the stochastic process U^o, is a random variable. Its expected value can be the time of the sign change of the suboptimal control of the forms (-1,+1) or (+1,-1), being then a deterministic function. Similarly, the case $x_0 \in (Q_- \cup Q_+)$ can be formalized.

However from the practical view point, closed-loop control structures are preferable. Similarly to the classical case, the time-optimal control U^o obtained above, can be defined also in a closed-loop system by the following formula:

$$U^o(X(\omega,t)) = \begin{cases} -1 & \text{if } X(\omega,t) \in R_- \\ +1 & \text{if } X(\omega,t) \in R_+ \; , \end{cases} \qquad (23)$$

and for $X(\omega,t) \in Q_- \cup Q_+$ a feedback controller function can be additionally defined without directly dependence on a random factor but only in a suboptimal way, e.g.:

$$U^s(X(\omega,t)) = \begin{cases} -a & \text{if } X(\omega,t) \in Q_- \\ +a & \text{if } X(\omega,t) \in Q_+ \; , \end{cases} \qquad (24)$$

where $0 < a \le 1$, however in practice the value of the parameter a should be closed to 1, e.g. $a = 0,95$ (the "probabilistic blur" of the switching area Q was exaggerated in Figure 1 for the sake of illustration, having been taken on $v_- = 0$ and $v_+ = 0,9$, whereas in the practice $v_- \cong 0,1$ and $v_+ \cong 0,3$). In case the simple solution above is not satisfactory related to the control quality required, then formula (24) can be precised in a more successful, but also more complicated way. Namely, after performing detailed analysis of the sensitivity of the control system to approximations of the model, one can use elements of statistical decision theory, where the loss function is connected with an extension of the time of reaching the target set if the change of control function sign was too late or too early.

EXAMPLE

First, some elements of statistical decision theory used in the following part, will be presented. The basic task of that theory is the separation of the "best" element from all possible decisions only on the basis of a probabilistic information about state of the nature, especially without the knowledge of its real state.

Let the following be given: the non-empty set N of possible states of the nature, the non-empty set D of possible decisions and the function:

$$l : D \times N \dashrightarrow \mathbb{R} \cup \{\pm\infty\} \qquad (25)$$

representing losses. Let $l_m : D \dashrightarrow \mathbb{R} \cup \{\pm\infty\}$ be a function of the so called minimax losses:

$$l_m(d) = \sup_{n \in N} l(d,n) \ ; \qquad (26)$$

if additionally on the set N the probability space (N,S,P) is defined, and for every $d \in D$ the integral $\int_N l(d,n)\, dP(n)$ exists, let $l_b : D \dashrightarrow \mathbb{R} \cup \{\pm\infty\}$ be a function of the so called Bayes losses:

$$l_b(d) = \int_N l(d,n)\ dP(n)\ . \qquad (27)$$

Then, every element $d_m \in D$ such that:

$$l_m(d_m) = \inf_{d \in D} l_m(d) \qquad (28)$$

is called a minimax decision and analogically, every element $d_b \in D$ such that:

$$l_b(d_b) = \inf_{d \in D} l_b(d) \qquad (29)$$

is called a Bayes decision.

Comparing the above decision rules, one can notice that in the case of the minimax rule it is not required to define the probability measure in the set N. Also, calculations are simpler in that case. The basic difference, however, occurs in the interpretation and it results straight from the form of the loss functions l_m and l_b: "pessimistic" minimax rule assumes the occurrence of the most unfavorable state of the nature, while "realistic" Bayes rule allows to receive the action which is the most favorable in the sense of the expected value.

Synthesis of the suboptimal feedback controller will be based on the following simplification: the realization of the stochastic process V will be substituted by the constant realization of the value equal to its average on the bounded interval L:

$$V(\omega,t) \ \longmapsto \ v(\omega) = \frac{1}{\|L\|} \int_L V(\omega,t)\ dt$$
$$\text{for almost all } \omega \in \Omega\ . \qquad (30)$$

One can easily notice that if the distribution connected with the stochastic process V has the particular property $v_- = v_+$, then the following equalities are true $K_{+-} = K_{++}$ and $K_{--} = K_{-+}$ (Fig. 1), so, the switching area Q_* is reduced to the some switching curve Q^*, which form is dependent on the above value $v_- = v_+$. But in the random object that value is unknown *a priori* with probability 1. Therefore, the value accepted in the feedback controller equations will be fixed on the basis of presented elements of decision theory. Namely, the value adopted in the feedback controller equations (symbolized by $W \in D$) will be treated as a decision, while the value of the random variable v occurring in the controlled object (symbolized by $w = v(\omega) \in N$) as a state of the nature. The values of the loss function are defined for $(W,w) \in D \times N$, and they are related to the time of reaching the target set, if in the feedback controller equations the value W was adopted, but in the object the value w is occurring. If the conditions which are determined by the parameter W value, have to be fulfilled under the danger of system damage, the minimax rule will be used for the purpose of its calculation; or else a more "flexible" the Bayes rule.

A detailed analysis of a sensitivity of the deterministic system created from the random object by fixing the value of the random variable $v(\omega) = w$, on the value of the parameter W occurring in the feedback controller equations, presented in [3], implies the following suggestions for the calculation of the parameter W value.

If over-regulations are not allowed in the controlled system, the selection of the parameter W value should be carried out on the basis of the minimax rule, assuming the infinite value of the loss function for $W > w$. However, if over-regulations can be allowed, it is worthwhile using the Bayes rule at real values of the loss function.

So, let v be a random variable whose distribution has the density function h, with a support of the form $[w_-,w_+] \subset (-1,1)$, and besides, in the set (w_-,w_+) let that function be continuous and fulfill the condition $h(w) > 0$. The loss function will be described by the formula:

$$l(W,w) = \begin{cases} -p\ (W - w) & \text{for } W - w \le 0 \\ \ \ r\ (W - w) & \text{for } W - w \ge 0\ , \end{cases} \qquad (31)$$

where p and r are real positive numbers or one of those elements equals ∞; let then $\infty \cdot 0 = 0$.

According to the above assumptions it should be accepted that $N = D = [w_-,w_+]$.

If $r = \infty$, then the minimax decision is [3]:

$$W = w_- \qquad (32)$$

At the real positive p and r, the Bayes

decision is the unique value of the parameter W satisfying the following equation [3]:

$$\int_{W_-}^{W} h(w)\, dw = \frac{r}{p + r} \tag{33}$$

The function of the suboptimal feedback controller, received with the application of the above algorithm using the Bayes rule, and representative trajectories occurring in the system, are illustrated in Figure 2.

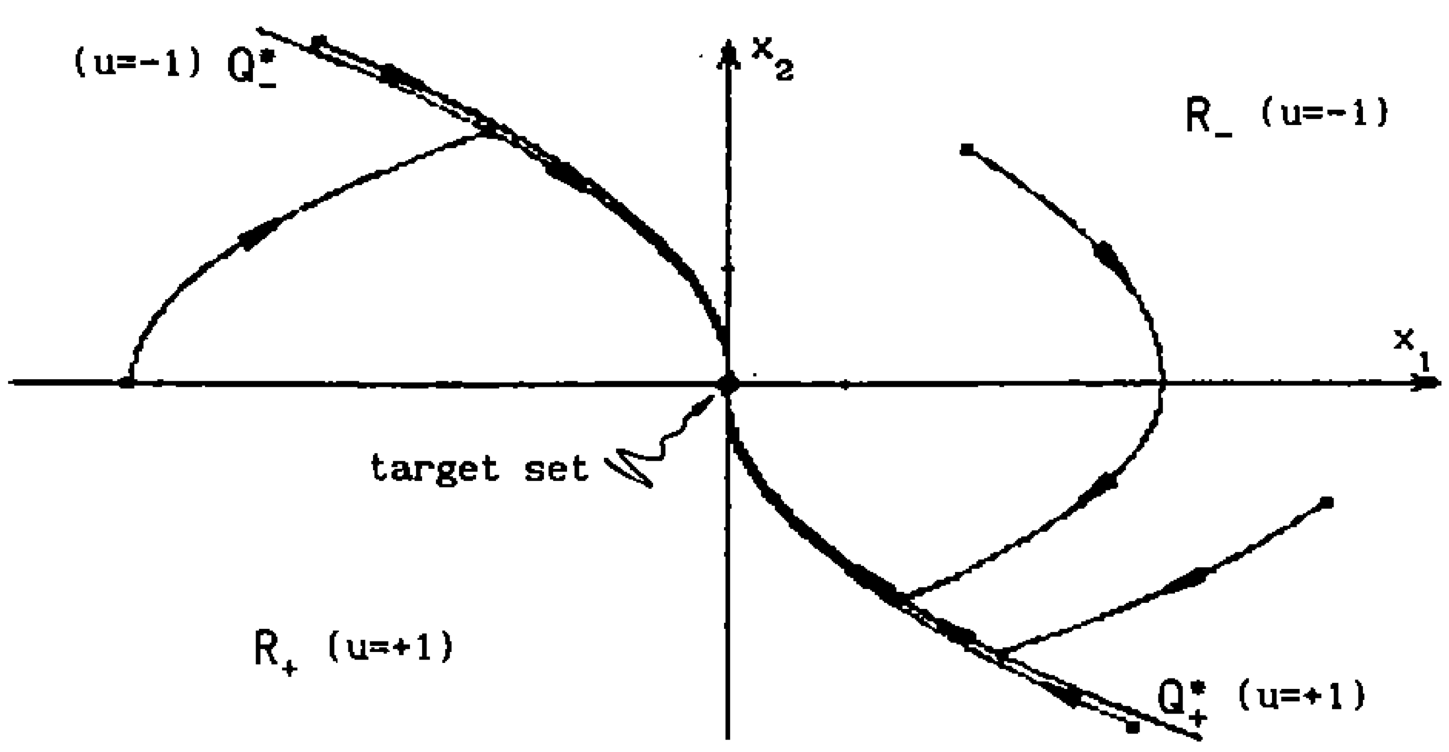

Fig. 2

Thanks to the theorem presented in this paper, the existence of almost certain F- and K-solutions (there is no almost certain C-solutions !) generated by the constructed feedback controller, has been proved in [3]. Because nonunique solutions can appear in the system (!) the proof has been based on the theory of measurable selectors.

The case where the target set differs from the origin of coordinates, can be considered analogically, though if the second coordinate of the target set differs from zero, it is much more complicated. Especially, the switching curve Q^* must be divided into three parts. The division points are the target set and the crossing point with the axis x_1. For every part, the value of the parameter W should be calculated in a different way [3]. Figure 3 illustrates the received suboptimal feedback controller function and representative trajectories.

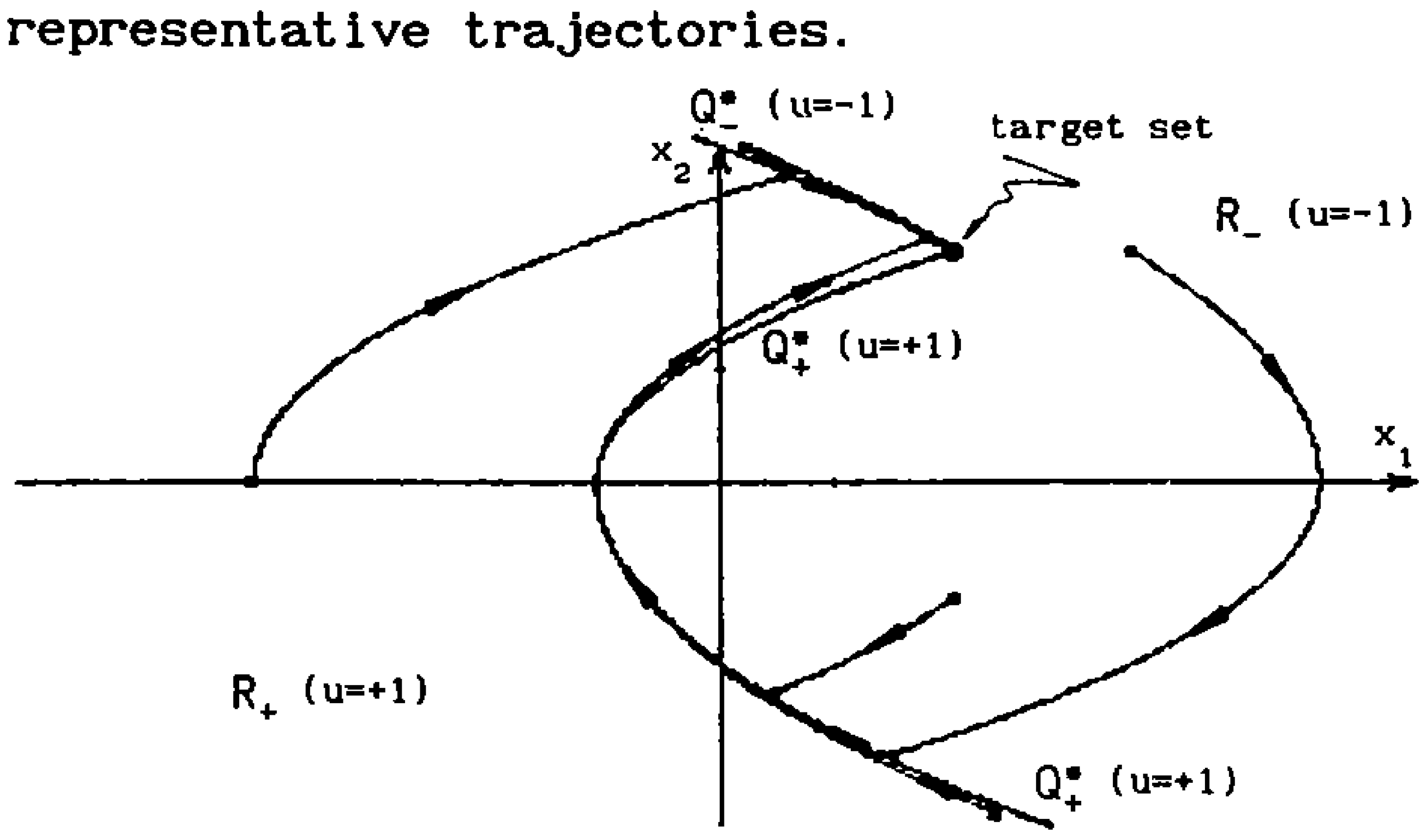

Fig. 3

A substantial number of completed empirical examinations [3] confirmed theoretical considerations and showed a correct operation of the suboptimal feedback controller proposed in this example. The target set was reached in every case with a precision in the size of 0,1-0,5% of the initial state norm, during a time which was close to minimal and was approaching it with the increase of the number of the data. If it was assumed that over-regulations were unacceptable, they did not occur during the control process. Only small dependence of received values of feedback controller parameters on initial and final conditions in previous recourses, which affected the data received, was recorded. However, with different characteristics (e.g. changing in time) of motion resistances, the differences can be considerable. So there appears a suggestion of an adaptive structure. Finally, the constructed controlling system turned out to be insensitive to the inaccuracy resulting from the identification process and to the perturbations occurring in the system; this is a very valuable property for random control systems. ■

To summarize, the introduction of the random factor to the description of system dynamics results in a considerable complexity of that task from a theoretical view point. However, practical effects can be significant. The theorem presented in this paper create a useful basis for the practical construction of time-optimal control structures, which can be easily realized thanks to the possibility of application of contemporary microprocessor systems, contributing to the increase of the effectiveness of many modern industrial devices.

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