Arity-Monotonic Extended Aggregation Operators

Marek Gągolewski*[1,2] and Przemysław Grzegorzewski[1,2]

[1] Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01-447 Warsaw, Poland
[2] Faculty of Mathematics and Information Science, Warsaw University of Technology
pl. Politechniki 1, 00-661 Warsaw, Poland
gagolews@ibspan.waw.pl, pgrzeg@ibspan.waw.pl

This is a revised version of the paper: Gągolewski M., Grzegorzewski P., *Arity-Monotonic Extended Aggregation Operators*, In: Hüllermeier E., Kruse R., Hoffmann F. (Eds.), *Information Processing and Management of Uncertainty in Knowledge-Based Systems*, CCIS **80**, Springer-Verlag, 2010, pp. 693–702.

Abstract. A class of extended aggregation operators, called impact functions, is proposed and their basic properties are examined. Some important classes of functions like generalized ordered weighted averaging (OWA) and ordered weighted maximum (OWMax) operators are considered. The general idea is illustrated by the Producer Assessment Problem which includes the scientometric problem of rating scientists basing on the number of citations received by their publications. An interesting characterization of the well known h-index is given.

Key words: aggregation, extended aggregation function, OWA, OWMax, h-index, scientometrics.

1 Introduction

Aggregation plays a central role in many areas of the human activity, including not only statistics, engineering, computer science or physics but also decision making, economy and social sciences. It appears always when the reasoning requires merging several values into a single one which may represent a kind of synthesis for all individual inputs. Such functions projecting multidimensional numerical space of input values into one dimension are generally called *aggregation operators*.

Apart from particular applications the theory of aggregation operators is a rapidly developing mathematical domain (we refer the reader to [6] for the recent state of the art monograph).

Classically, aggregation operators are usually considered for a fixed number of arguments. For some applications it may be to restrictive. We face such a situation in the so-called Producer Assessment Problem where given alternatives are rated not only with respect to the quality of delivered items but also to their productivity. As a typical example we may indicate the problem of rating scientists by their publications' citations.

* Corresponding author

This is the reason that the aggregation operators defined for arbitrary number of arguments are of interest. In the paper we propose the axiomatic approach to such a class of extended aggregation operators, called impact functions, and discuss some interesting properties of such functions for different arities. We also study the properties of the generalized versions of some well known classes of aggregation operators like generalized ordered weighted averaging (OWA) and ordered weighted maximum (OWMax) operators. It is worth noting that well-known Hirsch h-index turns out to be a particular example of the latter family.

2 Preliminaries

2.1 Basic Notation

We adopt the notational convention from the recent monograph [6].

Let $\mathbb{I} = [a, b]$ denote any nonempty closed interval of extended real numbers $\bar{\mathbb{R}} = [-\infty, \infty]$. In this paper we assume that $0 \cdot \infty = 0$. Unless stated otherwise, $n, m \in \mathbb{N}$.

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denote the set of all nonnegative integers. Moreover, let $[n] := \{1, 2, \dots, n\}$.

The set of all vectors of arbitrary length with elements in $\mathbb{I}$, i.e. $\bigcup_{n=1}^{\infty} \mathbb{I}^n$, will be denoted by $\mathbb{I}^{1,2,\dots}$.

Given any $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{I}^n$, we write $\mathbf{x} \le \mathbf{y}$ iff $(\forall i \in [n])$ $x_i \le y_i$. Moreover, $(n * x)$ is equivalent to $(x, x, \dots, x) \in \mathbb{I}^n$.

Let $x_{(i)}$ denote the ith-smallest value of $\mathbf{x} = (x_1, \dots, x_n)$. For simplicity of notation, we assume that $x_{(n+j)} := x_{(n)}$ for $j = 1, 2, \dots$.

For any $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{y} \in \mathbb{I}^m$ and any function f defined on $\mathbb{I}^{n+m}$ the notation $\mathsf{f}(\mathbf{x}, \mathbf{y})$ stands for $\mathsf{f}(x_1, \dots, x_n, y_1, \dots, y_m)$.

If $\mathsf{f} : X \to Y$ and $X' \subset X$ then a function $\mathsf{f}|_{X'} : X' \to Y$ such that $(\forall x \in X')$ $\mathsf{f}|_{X'}(x) = \mathsf{f}(x)$ is called a *restriction* of f to X'. Furthermore, if $\mathcal{F}$ is a family of functions mapping X to Y, then $\mathcal{F}|_{X'} := \{\mathsf{f}|_{X'} : \mathsf{f} \in \mathcal{F}\}$.

2.2 Aggregation Functions

Let us recall the notion of the aggregation function, which is often considered in the literature. Note that it is a particular sublass of the very broad family of aggregation operators. Here is a slightly modified version of the definition given in [6].

Definition 1. *An* ***aggregation function*** *in* $\mathbb{I}^n = [a, b]^n$ *is any function* $\mathsf{a}^{(n)} : \mathbb{I}^n \to \bar{\mathbb{R}}$ *which*

(nd) *is nondecreasing in each variable, i.e.* $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ $\mathbf{x} \le \mathbf{y} \Rightarrow \mathsf{a}^{(n)}(\mathbf{x}) \le \mathsf{a}^{(n)}(\mathbf{y})$,
(bl) *fulfills the lower boundary condition:* $\inf_{\mathbf{x} \in \mathbb{I}^n} \mathsf{a}^{(n)}(\mathbf{x}) = a$,
(bu) *fulfills the upper boundary condition:* $\sup_{\mathbf{x} \in \mathbb{I}^n} \mathsf{a}^{(n)}(\mathbf{x}) = b$.

Typical examples of aggregation functions are: sample minimum, maximum, arithmetic mean, median, and OWA operators. On the other hand, generally sample size, sum and constant function are not aggregation functions in the above sense.

It is worth noticing that axioms (nd) and (bl) imply $\mathsf{a}^{(n)}(n * a) = a$. We also have $\mathsf{a}^{(n)}(n * b) = b$ by (nd) and (bu).

From now on, let $\mathcal{E}(\mathbb{I})$ be the set of all functions $\mathsf{F} : \mathbb{I}^{1,2,\cdots} \to \bar{\mathbb{R}}$.

Now let us extend the class of aggregation functions to any number of arguments. Our definition agrees with the one given in [6]. Note that quite a different extension was proposed by Mayor and Calvo in [11].

Definition 2. *An **extended aggregation function** in $\mathbb{I}^{1,2,\cdots}$ is a function $\mathsf{A} \in \mathcal{E}(\mathbb{I})$, whose restriction $\mathsf{a}^{(n)} := \mathsf{A}|_{\mathbb{I}^n}$ to $\mathbb{I}^n$ for any $n \in \mathbb{N}$ is an aggregation function in $\mathbb{I}^n$.*

Note that any extended aggregation function may be regarded as a sequence $(\mathsf{a}^{(n)})_{n\in\mathbb{N}}$ of aggregation functions.

The set of all extended aggregation functions in $\mathbb{I}^{1,2,\cdots}$ will be denoted $\mathcal{E}_{\mathcal{A}}(\mathbb{I})$.

3 The Producer Assessment Problem and Impact Functions

Consider a **producer** (e.g. a writer, scientist, artist, craftsman) and a nonempty set of his **products** (e.g. books, papers, works, goods). Suppose that each product is given a rating (a single number in $\mathbb{I} = [a, b]$), where a denotes the lowest admissible rating. Here are some typical examples:

Producers	Products	Rates	Discipline
Scientists	Scientific articles	Number of citations	Scientometrics (see e.g. [7])
Scientific institutes	Scientists	The h-index	Scientometrics
Web servers	Web pages	Number of in-links	Webometrics
Artists	Paintings	Auction price	Auctions

Each possible state of a producer can be described by a point in $\mathbb{I}^{1,2,\cdots}$. The **Producer Assessment Problem** (or PAP for short) involves constructing and analyzing functions which can be used to rate producers. A family of such functions should take into account two following aspects of producer's quality:

1. the ability to make highly-rated products,
2. overall productivity.

Clearly, the first component can be described well by a very broad class of (extended) aggregation functions. However, in practice we are also interested in comparing entities with different productivity. Therefore, we need some sine qua non conditions for such assessing functions.

Definition 3. *An **impact function** in $\mathbb{I}^{1,2,\cdots}$ is a function $\mathsf{J} \in \mathcal{E}(\mathbb{I}), \mathbb{I} = [a, b]$ which is*

(nd) *nondecreasing in each variable:* $(\forall n)(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \le \mathbf{y} \Rightarrow \mathsf{J}(\mathbf{x}) \le \mathsf{J}(\mathbf{y})$,
(am) *arity-monotonic, i.e.* $(\forall n, m)(\forall \mathbf{x} \in \mathbb{I}^n)(\forall \mathbf{y} \in \mathbb{I}^m)\ \ \mathsf{J}(\mathbf{x}) \le \mathsf{J}(\mathbf{x}, \mathbf{y})$,

(sy) *symmetric, i.e.* $(\forall n)(\forall \mathbf{x} \in \mathbb{I}^n)\ (\forall \sigma \in \mathfrak{S}_{[n]})\ \ \mathsf{J}(x_1, \ldots, x_n) = \mathsf{J}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, *where* $\mathfrak{S}_{[n]}$ *denotes the set of all permutations of* $[n]$,

(bl) *fulfills the lower boundary condition:* $\inf_{\mathbf{x}\in\mathbb{I}^{1,2,\ldots}} \mathsf{J}(\mathbf{x}) = a$,

(wbu) *fulfills the weak upper boundary condition:* $\sup_{\mathbf{x}\in\mathbb{I}^{1,2,\ldots}} \mathsf{J}(\mathbf{x}) \le b$.

The set of all impact functions will be denoted by $\mathcal{E}_{\mathcal{I}}(\mathbb{I})$. Note that the set of requirements given in Def. 3 is a generalized version of the axiomatization proposed by Woeginger [15,16] for the scientometric impact indices (for other axiomatizations of the so-called bibliometric impact indices see [8,9,12]).

The property (am) expresses the intuition that in many applications of the PAP, the increase in production should not decrease the overall ranking. However, sometimes it be may be viewed as a weak point, because many aggregation operators are excluded.

Please note that impact functions are not necessarily aggregation functions (in the sense of Def. 1), because axiom (bu) is replaced by its weaker form (wbu). It is so because the upper boundary condition together with (am) seems too tight (if some $\mathsf{J} \in \mathcal{E}_{\mathcal{I}}(\mathbb{I})$ fulfills (bu), then — by $(\mathrm{bu}), (\mathrm{nd}), (\mathrm{am})$ and (sy) — $(\forall \mathbf{x} \in \mathbb{I}^n)(\exists i \in [n])$ $x_i = b$ implies $\mathsf{J}(x_1, \ldots, x_n) = b$).

4 Axiomatic Approach

4.1 Axiomatic Modeling

Below we discuss a set of properties which may be used to describe behavioral aspects of classes of impact functions. Formally, a **property** P of functions in $\mathcal{E}(\mathbb{I})$ is just a subset of $\mathcal{E}(\mathbb{I})$. We denote by $P_{(\mathrm{nd})}, P_{(\mathrm{am})}, \ldots$ the properties that appear in Def. 1 and Def. 3, i.e. some families of functions satisfying axioms $(\mathrm{nd}), (\mathrm{am}), \ldots$.

The concept of axiomatic modeling in decision making dates back as far as the works of Arrow [1] (impossibility theorem in the problem of social states ordering) and May [10] (group decision functions).

One approach considers a **characterization** of functions, i.e. a finite set of properties $P_1, \ldots, P_k \subseteq \mathcal{E}(\mathbb{I})$ such that $(!\exists f)\ f \in \bigcap_{i=1}^k P_i$. Moreover, that set should be minimal, i.e. $(\forall j \in [k])\ |\bigcap_{i=1, i\neq j}^k P_i| > 1$.

In the other approach a family of properties $\mathcal{P}$ that *seem* to be *sensible* from the *practical* point of view is of interest. Unfortunately quite often some of the properties $P_1, \ldots, P_l \in \mathcal{P}$ are contradictory, i.e. $P_1 \cap \cdots \cap P_l = \emptyset$. Therefore, in such a case there is no perfect (aggregation/impact) function that fulfill all imaginable properties.

4.2 Arity-Free Property

Generally, any two restrictions $\mathsf{A}|_{\mathbb{I}^n}$ and $\mathsf{A}|_{\mathbb{I}^m}$ of the extended aggregation function A, where $n \neq m$, are not necessarily related. However, here we are especially interested in properties which concern relations between restrictions of aggregation operators for different arities. Below we propose a formalism that concerns the above mentioned ideas.

Definition 4. *An **arity-free property** is any $P \subseteq \mathcal{E}(\mathbb{I})$ such that*

$$\left\{ \mathsf{F}|_{\mathbb{I}^m} : \mathsf{F} \in P,\ \mathsf{F}|_{\mathbb{I}^n} = \mathsf{f}^{(n)} \right\} = P|_{\mathbb{I}^m}$$

holds $(\forall n \neq m)$ $(\forall \mathsf{f}^{(n)} \in P|_{\mathbb{I}^n})$.

A family of all arity-free properties will be denoted by $\mathcal{P}_{\mathrm{af}}$. The other properties are called **arity-dependent**. Please notice that depending on the context we implicitly assume some fixed $\mathbb{I}$.

It can be seen easily that four of the axioms in Defs. 1 and 3 can be treated as arity-free properties, i.e. $P_{(\mathrm{nd})}, P_{(\mathrm{bl})}, P_{(\mathrm{bu})}, P_{(\mathrm{sy})} \in \mathcal{P}_{\mathrm{af}}$. However, in general $P_{(\mathrm{am})} \notin \mathcal{P}_{\mathrm{af}}$.

4.3 Ordering Property

Each function in $\mathcal{E}(\mathbb{I})$ implies a **ranking**, i.e. a linear (total) ordering relation in a set of producers' states. If valuation is expressed by a *single* numerical value such result can further be considered as a point-of-reference (e.g. for the author self-improving process). Therefore, it would be interesting to distinguish a class of properties that concern only the relation between the function values regardless of the particular values assumed by these functions.

Definition 5. *An **ordering property** is any $P \subseteq \mathcal{E}(\mathbb{I})$ such that the following condition*

$$\mathsf{F} \in P \Longrightarrow\ \mathsf{g} \circ \mathsf{F} \in P$$

holds for any nondecreasing function $\mathsf{g} : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$, where $\circ$ marks function composition, i.e. $(\mathsf{g} \circ \mathsf{F})(x) = \mathsf{g}(\mathsf{F}(x))$.

A family of all ordering properties will be denoted by $\mathcal{P}_{\mathrm{ord}}$. Note that $P_{(\mathrm{nd})}, P_{(\mathrm{am})}, P_{(\mathrm{sy})} \in \mathcal{P}_{\mathrm{ord}}$, but generally $P_{(\mathrm{bl})}$ and $P_{(\mathrm{bu})} \notin \mathcal{P}_{\mathrm{ord}}$.

Even though being obvious the following proposition is worth of explicit stating.

Proposition 1. *Any impact function cannot be defined by means of ordering properties only.*

Proof. Assume conversely that F is a unique function such that $\mathsf{F} \in P_1 \cap P_2 \cap \dots$, for some (possibly finite) sequence $P_1, P_2, \dots \subseteq \mathcal{P}_{\mathrm{ord}}$. Take any nondecreasing $\mathsf{g} : \bar{\mathbb{I}} \to \bar{\mathbb{R}}$ and let $\mathsf{F}' := \mathsf{g} \circ \mathsf{F}$. We have $\mathsf{F} \in P_1$ and $\mathsf{F}' \in P_1$. For any $i = 2, 3, \dots$ we have either $\mathsf{F}' \in P_1 \cap \dots \cap P_i$ or $P_1 \cap \dots \cap P_i = \emptyset$, which contradicts our assumption. $\Box$

Proposition 1 can also be formulated as follows: Any intersection of ordering properties is also an ordering property.

4.4 Further results

The following lemma allows to check efficiently whether a non-decreasing function is arity-monotonic.

Lemma 1. *For any* $\mathsf{F} \in P_{(\mathrm{nd})}$ *we have*

$$\mathsf{F} \in P_{(\mathrm{am})} \iff (\forall \mathbf{x} \in \mathbb{I}^{1,2,\cdots})\ \mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{x}, \min \mathbb{I}).$$

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Fix $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{y} \in \mathbb{I}^m$ for some n and m. We have $\mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{x}, \min \mathbb{I}) \le \mathsf{F}(\mathbf{x}, 2 * \min \mathbb{I}) \le \cdots \le \mathsf{F}(\mathbf{x}, m * \min \mathbb{I}) \le \mathsf{F}(\mathbf{x}, \mathbf{y})$ by (nd), since $(\mathbf{x}, m * \min \mathbb{I}) \le (\mathbf{x}, \mathbf{y})$. □

In addition to axiom (am) some other reasonable arity-dependent conditions could be considered.

It can sometimes be justifiable to treat the value $a = \min \mathbb{I}$ as the "minimal **admissible** quality". Adding new products with such rate (negligible elements) should not affect the overall ranking.

Definition 6. *We say that a function* $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ *is* ***zero-insensitive*** *(denoted* $\mathsf{F} \in P_{(\mathrm{zi})}$*) iff* $\mathsf{F}(\mathbf{x}, a) = \mathsf{F}(\mathbf{x})$ *for any* $\mathbf{x} \in \mathbb{I}^{1,2,\cdots}$.

We have $P_{(\mathrm{zi})} \cap P_{(\mathrm{nd})} \subseteq P_{(\mathrm{am})} \cap P_{(\mathrm{nd})}$ (by Lemma 1) and $P_{(\mathrm{zi})} \in \mathcal{P}_{ord}$.

Definition 7. *We say that a function* $\mathsf{F} \in P_{(\mathrm{bl})} \cap P_{(\mathrm{wbu})}$ *satisfies* ***condition*** $(\mathrm{s}-)$ *(denoted* $\mathsf{F} \in P_{(\mathrm{s}-)}$*) iff* $(\forall \mathbf{x} \in \mathbb{I}^{1,2,\cdots})\ (\forall y \in \mathbb{I})\ \ y \le \mathsf{F}(\mathbf{x}) \Longrightarrow \mathsf{F}(\mathbf{x}, y) \le \mathsf{F}(\mathbf{x})$.

Note that if $\mathsf{F} \in \mathcal{E}_{\mathcal{I}}(\mathbb{I})$ then $\mathsf{F} \in P_{(\mathrm{s}-)}$ iff $(\forall \mathbf{x} \in \mathbb{I}^{1,2,\cdots})\ \mathsf{F}(\mathbf{x}, \mathsf{F}(\mathbf{x})) = \mathsf{F}(\mathbf{x})$. We may also see that $\mathsf{F} \in P_{(\mathrm{s}-)} \Rightarrow \mathsf{F} \in P_{(\mathrm{zi})}$.

Definition 8. *We say that a function* $\mathsf{F} \in P_{(\mathrm{bl})} \cap P_{(\mathrm{wbu})}$ *satisfies* ***condition*** $(\mathrm{s}+)$ *(denoted* $\mathsf{F} \in P_{(\mathrm{s}+)}$*) iff* $(\forall y \in \mathbb{I}, y > a)\ \mathsf{F}(y) > a$ *and* $(\forall \mathbf{x} \in \mathbb{I}^{1,2,\cdots})\ (\forall y \in \mathbb{I})\ y > \mathsf{F}(\mathbf{x}) \Longrightarrow \mathsf{F}(\mathbf{x}, y) > \mathsf{F}(\mathbf{x})$.

Please, notice that both $P_{(\mathrm{s}+)}$ and $P_{(\mathrm{s}-)} \notin \mathcal{P}_{\mathrm{ord}}$.

5 Exemplary Impact Functions

Below we examine two classes of important and interesting aggregation operators: ordered linear combination and ordered conditional maximum which generalize OWA and OWMax operators, respectively. From now on, let $\mathbb{I} = [0, \infty]$.

5.1 Ordered Linear Combination

Definition 9. *A* ***triangle of coefficients*** *is a sequence* $\triangle = (c_{i,n} \in \bar{\mathbb{R}} : i \in [n], n \in \mathbb{N})$.

Such object can be represented by

$$\begin{array}{llll} c_{1,1} & & & \\ c_{1,2} & c_{2,2} & & \\ c_{1,3} & c_{2,3} & c_{3,3} & \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Definition 10. *Given arbitrary triangle of coefficients $\triangle = (c_{i,n})_{i\in[n],n\in\mathbb{N}}$ the* ***ordered linear combination*** *associated with $\triangle$ is a function $\mathsf{OLC}_\triangle \in \mathcal{E}(\mathbb{I})$ such that for any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{I}^{1,2,\ldots}$*

$$\mathsf{OLC}_\triangle(\mathbf{x}) = \sum_{i=1}^{n} c_{i,n}\, x_{(n-i+1)}, \tag{1}$$

where $x_{(n-i+1)}$ denotes the ith-largest value of $\mathbf{x}$.

A special case of OLC is the class of *ordered weighted averaging functions* (OWA, introduced in [17]) with $\triangle = (w_{i,n})_{i\in[n],n\in\mathbb{N}}$ such that $(\forall n)\ \sum_{j=1}^{n} w_{j,n} = 1$ and $w_{i,n} \in [0,1]$ for $i \in [n]$. So defined $\triangle$ is called a *weighting triangle* (see e.g. [3,4]) as an extension of OWA for input vectors of arbitrary length).

Lemma 2. *For $\mathbb{I} = [a,b]$, $a \geq 0$, any $n \in \mathbb{N}$ and given $\mathbf{c}, \mathbf{c}' \in [0,\infty]^n$ we have*

$$(\forall \mathbf{x} \in \mathbb{I}^n)\ \sum_{i=1}^{n} c_i\, x_{(n-i+1)} \geq \sum_{i=1}^{n} c'_i\, x_{(n-i+1)} \iff (\forall k \in [n])\ \sum_{i=1}^{k} c_i \geq \sum_{i=1}^{k} c'_i. \tag{2}$$

Proof. $\sum_{i=1}^{n} c_i\, x_{(n-i+1)} - \sum_{i=1}^{n} c'_i\, x_{(n-i+1)} = \sum_{i=1}^{n-1}(c_i - c'_i)\left(x_{(n-i+1)} - x_{(1)}\right) + x_{(1)} \sum_{i=1}^{n}(c_i - c'_i) = \sum_{k=1}^{n}\left(x_{(n-k+1)} - x_{(n-k)}\right)\sum_{i=1}^{k}(c_i - c'_i) = (*)$, where $x_{(0)} = 0$. For every $j \in [n]$ we have $x_{(n-j+1)} - x_{(n-j)} \geq 0$ because $\min \mathbb{I} \geq 0$. Therefore $(*) \geq 0$ holds for all $\mathbf{x}$ iff $\sum_{i=1}^{k}(c_i - c'_i) \geq 0$ for every $k \in [n]$. $\square$

It can be seen easily that we have equality at the left side of (2) iff $(\forall i \in [n])\ c_i = c'_i$.

Proposition 2. *For $\mathbb{I} = [0,\infty]$ and any $\triangle = (c_{i,n})_{n\in\mathbb{N},i\in[n]}$*

a) $\mathsf{OLC}_\triangle \in \mathcal{E}_{\mathcal{A}}(\mathbb{I})$ *iff* $(\forall n)\ (\forall i \in [n])\ c_{i,n} \geq 0$ *and* $(\exists j \in [n])\ c_{j,n} > 0$.
b) $\mathsf{OLC}_\triangle \in \mathcal{E}_{\mathcal{I}}(\mathbb{I})$ *iff* $(\forall n)\ (\forall i \in [n])\ c_{i,n} \geq 0$ *and* $\sum_{j=1}^{i} c_{j,n+1} \geq \sum_{j=1}^{i} c_{j,n}$.

An easy proof based on Lemma 1 and 2 is omitted. Axiom (nd) is fulfilled due to the condition $c_{i,n} \geq 0$. Such $\mathsf{OLC}_\triangle$ is called an ordered conical combination.

Note that for an interval $\mathbb{I}' = [a,b]$, where $a \geq 0$ and $b < \infty$, under (nd), axioms (bl) and (bu) hold if and only if $\mathsf{OLC}_\triangle$ is an OWA (a.k.a. ordered convex combination). In that case the only aggregation function which is an impact function is the sample maximum $\mathsf{Max}(x_1, \ldots, x_n) := x_{(n)}$.

Let us consider other properties mentioned in Sec. 4.4.

Lemma 3. *For $\mathbb{I} = [0,\infty]$ and any $\triangle = (c_{i,n})_{n\in\mathbb{N},i\in[n]}$ such that $\mathsf{OLC}_\triangle \in \mathcal{E}_{\mathcal{I}}(\mathbb{I})$ the following holds.*

a) $\mathsf{OLC}_\triangle \in P_{(\mathrm{zi})}$ *iff* $(\forall n)(\forall i \in [n])\ c_{i,n+1} = c_{i,n} \geq 0$.
b) $\mathsf{OLC}_\triangle \in P_{(\mathrm{s-})}$ *iff* $(\exists j \in \mathbb{N})\ (\forall n)\ c_{j,n} = q$ *for* $j \leq n$ *and* $(\forall i \in [n], i \neq j)\ c_{i,n} = 0$, *where* $q \in (0,1]$.
c) $\mathsf{OLC}_\triangle \in P_{(\mathrm{s+})} \cap P_{(\mathrm{zi})}$ *iff* $c_{1,1} > 0$, $(\forall n)\ c_{i,n+1} = c_{i,n} \geq 0$ *and if* $\sum_{i=1}^{n} c_{i,n} < 1$ *then* $c_{n+1,n+1} > 0$.

Proof (Sketch). a) Obvious.

b) Let us fix n. We have to consider 3 cases. 1. If $c_{\cdot,n} = (n * 0)$ then condition (s+) holds iff $c_{\cdot,n+1} = (n * 0, q)$ for some $q \geq 0$; 2. If $c_{\cdot,n} = ((n-1) * 0, q)$ for some $q > 0$ then (s+) (for any $\mathbf{x}$) iff $q \in (0,1]$ and $c_{\cdot,n+1} = ((n-1) * 0, q, 0)$; 3. If $c_{\cdot,n} = ((i-1) * 0, q, (n-i) * 0)$ for some $q \geq 0$ and $i \in [n-1]$ then (s+) iff $c_{\cdot,n} = ((i-1) * 0, q, (n-i+1) * 0)$.

c) Note that under (zi), the triangle of coefficients may be reduced to a sequece $\mathbf{c} = (c_1, c_2, \dots)$. Then $\mathsf{OLC}_{\triangle}(\mathbf{x}) = \mathsf{OLC}_{\mathbf{c}}(\mathbf{x}) = \sum_{i=1}^{n} c_i x_{(n-i+1)}$.

We have $\mathsf{OLC}_{\mathbf{c}}(0) = 0$. Assume that $c_1 = 0$ and $\mathsf{OLC}_{\mathbf{c}} \in P_{(\mathrm{s}+)}$. Then for each $\varepsilon > 0$ it holds $\mathsf{OLC}_{\mathbf{c}}(\varepsilon, 0) = 0 \not> \mathsf{OLC}_{\mathbf{c}}(0)$, a contradiction, therefore $c_1 > 0$.

Fix n. We consider 2 cases.

- Let $\sum_{i=1}^{n} c_i < 1$ and $\mathbf{x} = (n * d)$ for $d \in (0, \infty)$. Then $\mathsf{OLC}_{\mathbf{c}}(\mathbf{x}) < d$. So it is necessary that $c_{n+1} > 0$. Moreover, by induction, $(\forall i \leq n)\ c_i > 0$. It is easily seen that it is also a sufficient condition for (s+).
- Let $\sum_{i=1}^{n} c_i \geq 1$ and $j = \min\{j : \sum_{i=1}^{j} c_i\}$. We have $(\forall k \in [j])\ c_k > 0$. Take any $\mathbf{x} \in \mathbb{I}^n$, $\varepsilon > 0$ and $\mathbf{y} = (x_{(n)}, \dots, x_{(2)}, \mathsf{OLC}_{\mathbf{c}}(\mathbf{x}) + \varepsilon)$. As $\mathsf{OLC}_{\mathbf{c}}(\mathbf{x}) + \varepsilon > x_{(n-j+1)}$, then $\mathsf{OLC}_{\mathbf{c}}(\mathbf{x}) < \mathsf{OLC}_{\mathbf{c}}(\mathbf{y}) \leq \mathsf{OLC}_{\mathbf{c}}(\mathbf{x}, \mathsf{OLC}_{\mathbf{c}}(\mathbf{x}) + \varepsilon)$ for all $c_{n+1,n+1} \geq 0$.

□

Note that both (s−) and (s+) hold if and only if $c_{1,n} = 1$ and $c_{j,n} = 0$ for any n and $j > 1$, i.e. $\mathsf{OLC}_{\triangle}$ is the sample maximum.

5.2 Ordered Conditional Maximum

Definition 11. *The* ***ordered conditional maximum*** *associated with a triangle of coefficients* $\triangle = (c_{i,n} \in \mathbb{I} : i \in [n], n \in \mathbb{N})$ *is a function* $\mathsf{OCM}_{\triangle} \in \mathcal{E}(\mathbb{I})$ *such that*

$$\mathsf{OCM}_{\triangle}(\mathbf{x}) = \bigvee_{i=1}^{n} c_{i,n} \wedge x_{(n-i+1)} \tag{3}$$

for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{I}^{1,2,\dots}$.

A particular case of OCM is the ordered weighted maximum operator (OWMax) introduced in [5], defined for $\mathbb{I} = [0,1]$ and such that $\bigvee_{i=1}^{n} c_{i,n} = 1$ and $c_{i,n} \in [0,1]$.

Furthermore, OCM also generalizes the well-known h-index (see [14]). The h-index was originally defined by Hirsch [7] for ratings in $\mathbb{N}_0$ as a function h such that

$$\mathsf{h}(x_1, \dots, x_n) = \max\{i = 0, \dots, n : x_{(n-i+1)} \geq i\}.$$

It was proposed as a method of assessing scientific merit of individual researchers by means of the number of citations received by their scientific papers. Its popularity possibly arose from an appealing interpretation: one has h-index of, say H, if H of his papers gained at least H and the remaining $n - H$ papers at most H citations. Interestingly, a similar object was earlier defined in the context of Bonferroni-type multiple significance testing (see e.g. [2]).

Lemma 4. *If* $\triangle = (c_{i,n})_{i\in[n],n\in\mathbb{N}}$ *such that* $c_{i,n} = i$ *for* $n \in \mathbb{N}$ *and* $i \in [n]$ *then for any* $m \in \mathbb{N}$, $x_1, \dots, x_m \in \mathbb{I} \cap \mathbb{N}_0$,

$$\mathsf{OCM}_\triangle(x_1, \dots, x_m) = \mathsf{h}(x_1, \dots, x_m).$$

Proof. Let $H = \max\{i : x_{(n-i+1)} \geq i\}$. We have $\bigvee_{i=1}^{H} i \wedge x_{(n-i+1)} = H$ and $\bigvee_{i=H+1}^{n} i \wedge x_{(n-i+1)} = x_{(n-H)} < H + 1$. However, since $x_{(n-H)} \in \mathbb{N}_0$, then $x_{(n-H)} \leq H$ and therefore $\bigvee_{i=1}^{n} i \wedge x_{(n-i+1)} = H$. □

Note that we have $\mathsf{h}(2, 1.5) = 1$ but $\mathsf{OCM}_\triangle(2, 1.5) = 1.5$. Generally, for arbitrary $\mathbf{x}$, $\mathsf{OCM}_\triangle(\mathbf{x}) = \max\{H, x_{(n-H)}\} \in [H, H + 1)$.

Lemma 5. *For any* $\mathbb{I}$ *and any* $n \in \mathbb{N}$*, given* $\mathbf{c}, \mathbf{c}' \in \mathbb{I}^n$ *we have*

$$(\forall \mathbf{x} \in \mathbb{I}^n)\ \bigvee_{i=1}^{n} c_i \wedge x_{(n-i+1)} \geq \bigvee_{i=1}^{n} c'_i \wedge x_{(n-i+1)} \Longleftrightarrow (\forall k \in [n])\ \bigvee_{i=1}^{k} c_i \geq \bigvee_{i=1}^{k} c'_i. \quad (4)$$

The proof is omitted. Note that if $K = \{k = 2, 3, \dots, n : c_k \leq \bigvee_{i=1}^{k-1} c_i\}$ then $\bigvee_{i=1}^{n} c_i \wedge x_{(n-i+1)} = \bigvee_{i=1, i\notin K}^{n} c_i \wedge x_{(n-i+1)}$. Additionally, it is easily seen that we have equality at the left side of (4) iff $(\forall k \in [n])\ \bigvee_{i=1}^{k} c_i = \bigvee_{i=1}^{k} c'_i$.

Proposition 3. *For any* $\mathbb{I} = [a, b]$ *and any* $\triangle = (c_{i,n})_{i\in[n],n\in\mathbb{N}}, c_{i,n} \in \mathbb{I}$

a) $\mathsf{OCM}_\triangle \in \mathcal{E}_{\mathcal{A}}(\mathbb{I})$ *iff* $(\forall n)\ (\exists j \in [n])\ c_{j,n} = b$.
b) $\mathsf{OCM}_\triangle \in \mathcal{E}_{\mathcal{I}}(\mathbb{I})$ *iff* $(\forall n)\ (\forall i \in [n])\ c_{i,n} \geq a$ *and* $\bigvee_{j=1}^{i} c_{j,n+1} \geq \bigvee_{j=1}^{i} c_{j,n}$.

The proof is omitted. Let us consider other properties.

Lemma 6. *For any* $\mathbb{I} = [a, b]$ *and any* $\triangle = (c_{i,n})_{i\in[n],n\in\mathbb{N}}$ *such that* $\mathsf{OCM}_\triangle \in \mathcal{E}_{\mathcal{I}}(\mathbb{I})$ *the following holds:*

a) $\mathsf{OCM}_\triangle \in P_{(\mathrm{zi})}$ *iff* $(\forall n)(\forall i \in [n])\ \bigvee_{j=1}^{i} c_{j,n+1} = \bigvee_{j=1}^{i} c_{j,n}$.
b) $\mathsf{OCM}_\triangle \in P_{(\mathrm{s}-)}$ *iff* $\mathsf{OCM}_\triangle \in P_{(\mathrm{zi})}$.
c) $\mathsf{OCM}_\triangle \in P_{(\mathrm{s}+)}$ *iff* $c_{1,1} > a$ *and* $(\forall n)$ *if* $c_{1,n} < b$ *then* $c_{1,n+1} > \bigvee_{i\in[n],c_{i,n}<b} c_{i,n}$.

Proof (Sketch). a) It follows from the remark to Lemma 5.

b) Let us fix n. We should only show that $(\forall i \in [n])\ \bigvee_{j=1}^{i} c_{j,n} = \bigvee_{j=1}^{i} c_{j,n+1}$ implies $\mathsf{OCM}_\triangle \in (\mathrm{s}-)$. Let $\mathsf{OCM}_\triangle(\mathbf{x}) = c_{j,n} \wedge x_{(n-j+1)}$ for some j. But as $\mathsf{OCM}_\triangle(\mathbf{x}) \leq x_{(n-j+1)}$ and $x_{(n-j+1)} \wedge \bigvee_{i=j+1}^{n+1} c_{i,n+1} \leq x_{(n-j+1)}$, it holds $\mathsf{OCM}_\triangle(\mathbf{x}, \mathsf{OCM}_\triangle(\mathbf{x})) = c_{j,n+1} \wedge x_{(n-j+2)} = c_{j,n} \wedge x_{(n-j+1)} = \mathsf{OCM}_\triangle(\mathbf{x})$.

c) Let us fix n and let $c_{j,n} = \bigvee_{i\in[n],c_{i,n}<b} c_{i,n}$ for some $j \in [n]$. Take $\mathbf{x} = (n{*}c_{j,n})$. Then for any $\varepsilon > 0$ $\mathsf{OCM}_\triangle(\mathbf{x}, c_{j,n} + \varepsilon) > c_{j,n} = \mathsf{OCM}_\triangle(\mathbf{x})$ iff $c_{1,n+1} > c_{j,n}$. Now take any $\mathbf{y} \in \mathbb{I}^n$. Let $\mathsf{OCM}_\triangle(\mathbf{y}) = c_{j,n} \wedge y_{(n-j+1)}$ for some j. Then $c_{j,n} \wedge y_{(n-j+1)} < c_{1,n+1} \wedge ((c_{j,n} \wedge y_{(n-j+1)}) + \varepsilon)$, which completes the proof. □

All $\mathsf{OCM}_\triangle$ satisfying both conditions $(\mathrm{s}-)$ and $(\mathrm{s}+)$ are equivalent to the sample maximum (when $c_{1,1} = b$). The extended h-index is an impact function satisfying (zi).

6 Conclusions

In the paper we have considered a class of aggregation operators and discussed their basic properties. The particular attention has been directed to remarkable classes of such functions, i.e. ordered linear combination and ordered conditional maximum operators, which generalize OWA and OWMax operators, respectively. However, extensions of many other classes of aggregation operators would be interesting too.

The problem of ratings based on citations was mentioned to illustrate the need and the importance of such extensions of the aggregation operators. Nowadays the aforementioned h-index is probably the best known scientometric tool. However, many other interesting bibliometric indices exist in the literature and they surely could be also characterized in the framework of the impact functions. This is the topic of our further research.

Finally, we want to stress that the suggested generalization of aggregation operators might have applications not only in scientometrics or — generally — the Producer Assessment Problem but in many other fields. However, one has to be conscious that aggregation performed in some special areas may potentially require other particular requirements that should be expressed by different axioms.

References

1. Arrow, K.J.: A difficulty in the concept of social welfare. Journal of Political Economy 58(4), 328–346 (1950)
2. Benjamini, Y., Hochberg, Y.: Controlling false discovery rate: a practical and powerful approach to multiple testing. Journal of the Royal Statistical Society. Series B 57(1), 289–300 (1995)
3. Calvo, T., Kolesarova, A., Komornikova, M., Mesiar, R.: Aggregation operators: properties, classes and construction methods. In: Calvo, T., Mayor, G., Mesiar, R. (eds.) Aggregation Operators. New Trends and Applications, Studies in Fuzziness and Soft Computing, vol. 97, pp. 3–104. Physica-Verlag, New York (2002)
4. Calvo, T., Mayor, G.: Remarks on two types of extended aggregation functions. Tatra Mountains Mathematical Publications 16, 235–253 (1999)
5. Dubois, D., Prade, H., Testemale, C.: Weighted fuzzy pattern matching. Fuzzy Sets and Systems 28, 313–331 (1988)
6. Grabisch, M., Pap, E., Marichal, J.L., Mesiar, R.: Aggregation Functions. Cambridge (2009)
7. Hirsch, J.E.: An index to quantify individual's scientific research output. Proceedings of the National Academy of Sciences 102(46), 16569–16572 (2005)
8. Marchant, T.: An axiomatic characterization of the ranking based on the h-index and some other bibliometric rankings of authors. Scientometrics 80(2), 325–342 (2009)
9. Marchant, T.: Score-based bibliometric rankings of authors. Journal of the American Society for Information Science and Technology 60(6), 1132–1137 (2009)
10. May, K.O.: A set of independent necessary and sufficient conditions for simple majority decision. Econometrica 20(4), 680–684 (1952)
11. Mayor, G., Calvo, T.: On extended aggregation functions. In: Proc. IFSA 1997. vol. 1, pp. 281–285. Academia, Prague (1997)
12. Palacios-Huerta, I., Volij, O.: The measurement of intellectual influence. Econometrica 72(3), 963–977 (2004)

13. Torra, V. (ed.): Information Fusion in Data Mining, Studies in Fuzziness and Soft Computing, vol. 123. Springer-Verlag (2003)
14. Torra, V., Narukawa, Y.: The h-index and the number of citations: two fuzzy integrals. IEEE Transactions on Fuzzy Systems 16(3), 795–797 (2008)
15. Woeginger, G.J.: An axiomatic analysis of Egghe's g-index. Journal of Informetrics 2(4), 364–368 (2008)
16. Woeginger, G.J.: An axiomatic characterization of the Hirsch-index. Mathematical Social Sciences 56(2), 224–232 (2008)
17. Yager, R.R.: On ordered weighted averaging aggregation operators in multictriteria decision making. IEEE Transactions on Systems, Man, and Cybernetics 18(1), 183–190 (1988)

Please cite this paper as: Gągolewski M., Grzegorzewski P., *Arity-Monotonic Extended Aggregation Operators*, In: Hüllermeier E., Kruse R., Hoffmann F. (Eds.), *Information Processing and Management of Uncertainty in Knowledge-Based Systems*, CCIS **80**, Springer-Verlag, 2010, pp. 693-702.