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Weak solutions to 3–D Cahn-Hilliard system in elastic solids

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Abstract. In this paper we prove the existence and some time regularity of weak solutions to a three-dimensional (3–D) Cahn-Hilliard system coupled with nonstationary elasticity. Such nonlinear parabolic-hyperbolic system arises as a model of phase separation in deformable alloys. The regularity result is based on the analysis of time differentiated problem by means of the Faedo-Galerkin method. The obtained regularity provides a first step to the proof of strong solvability of the problem to be presented in a forthcoming paper [PawZaj06c].
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1. Introduction

The present paper is concerned with the existence and regularity of weak solutions to a three-dimensional (3–D) Cahn-Hiliard system coupled with nonstationary elasticity. Such system arises as a model of phase separation in a binary deformable alloy quenched below a critical temperature. The problem under consideration has the following form:

$$\text{(1.1)}\qquad \begin{aligned} &\boldsymbol{u}_{tt} - \nabla\cdot W_{,\varepsilon}(\varepsilon(\boldsymbol{u}),\chi) = \boldsymbol{b} && \text{in } \Omega^T = \Omega\times(0,T),\\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega,\\ &\boldsymbol{u} = \boldsymbol{0} && \text{in } S^T = S\times(0,T), \end{aligned}$$

$$\text{(1.2)}\qquad \begin{aligned} &\chi_t - \nabla\cdot M\nabla\mu = 0 && \text{in } \Omega^T,\\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega,\\ &\boldsymbol{n}\cdot M\nabla\mu = 0 && \text{on } S^T, \end{aligned}$$

$$\text{(1.3)}\qquad \begin{aligned} &\mu = -\nabla\cdot\boldsymbol{\Gamma}\nabla\chi + \psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi) && \text{in } \Omega^T,\\ &\boldsymbol{n}\cdot\boldsymbol{\Gamma}\nabla\chi = 0 && \text{on } S^T. \end{aligned}$$

Here $\Omega\subset\mathbb{R}^3$ is a bounded domain with a smooth boundary S, occupied by a body in a reference configuration with constant mass density $\varrho = 1$; $\boldsymbol{n}$ is the unit outward normal to S, and $T>0$ is an arbitrary fixed time. The body is a binary $a-b$ alloy.

The unknowns are the fields $\boldsymbol{u},\chi$ and μ, where $\boldsymbol{u}:\Omega^T\to\mathbb{R}^3$ is te *displacement vector* $\chi:\Omega^T\to\mathbb{R}$ is the *order parameter* (phase ratio) and $\mu:\Omega^T\to\mathbb{R}$ is the chemical potential difference between the components, shortly referred to as the *chemical potential.*

The second order tensor

$$\varepsilon = \varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$$

denotes the linearized strain tensor.

In case of a binay a–b alloy the order parameter is related to the volumetric fraction of one of the two phases, characterized by different crystalline structures of the components, e.g. $\chi=-1$ is identified with the phase a and $\chi = 1$ with the phase b.

The function $W(\varepsilon(\boldsymbol{u}),\chi)$ denotes the elastic energy defined by

$$\text{(1.4)}\qquad W(\varepsilon(\boldsymbol{u}),\chi) = \frac{1}{2}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi))\cdot \boldsymbol{A}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi)).$$

The corresponding derivatives

$$W_{,\varepsilon}(\varepsilon(\boldsymbol{u}),\chi) = \boldsymbol{A}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi)),$$

and

$$W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi) = -\bar{\varepsilon}'(\chi)\cdot \boldsymbol{A}(\varepsilon(\boldsymbol{u}) - \bar{\varepsilon}(\chi))$$

represent respectively the stress tensor and the elastic part of the chemical potential. The fourth oredr tensor $\boldsymbol{A} = (A_{ijkl})$ denotes a constant elasticiy tensor given by

$$\varepsilon(\boldsymbol{u}) \mapsto \boldsymbol{A}\varepsilon(\boldsymbol{u}) = \bar{\lambda} tr\varepsilon(\boldsymbol{u})\boldsymbol{I} + 2\bar{\mu}\varepsilon(\boldsymbol{u}), \tag{1.5}$$

where $\boldsymbol{I} = (\delta_{ij})$ is the identity tensor, and $\bar{\lambda}, \bar{\mu}$ are the Lamé constants with values within the elasticity range (see Section 2).The form (1.5) refers to the isotropic, homogeneous medium with the same elastic properties of the phases.

The second order tensor $\bar{\varepsilon}(\chi)$ denotes the eigenstrain, i.e. the stress free strain corresponding to the phase ratio χ, defined by

$$\bar{\varepsilon}(\chi) = (1 - z(\chi))\bar{\varepsilon}_a + z(\chi)\bar{\varepsilon}_b, \tag{1.6}$$

with $\bar{\varepsilon}_a, \bar{\varepsilon}_b$ denoting costant eigenstrains of the phases a, b, and $z : \mathbb{R} \to [0,1]$ being a sufficiently smooth interpolation function (called shape function) satisfying

$$z(\chi) = 0 \quad \text{for} \quad \chi \le -1 \quad \text{and} \quad z(\chi) = 1 \quad \text{for} \quad \chi \ge 1. \tag{1.7}$$

Furthermore, the function $\psi : \mathbb{R} \to \mathbb{R}$ denotes the chemical energy of the system at zero stress. This function depends on temperature and is convex above a critical temperature and a nonconvex for temperatures less than the critical one. Here we assume it in the simplest double-well form

$$\psi(\chi) = \frac{1}{4}(1 - \chi^2)^2 \tag{1.8}$$

with two minima at $\chi = -1$ and $\chi = 1$.

The second order tensors $\boldsymbol{M} = (M_{ij})$ and $\boldsymbol{\Gamma} = (\Gamma_{ij})$ represent respectivly the mobility matrix and the interfacial energy matrix. For simplicity, we shall confine ourselves to the isotropic, homogeous situation assuming that

$$\boldsymbol{M} = M\boldsymbol{I}, \quad \boldsymbol{\Gamma} = \Gamma\boldsymbol{I}, \quad M = \Gamma = 1, \tag{1.9}$$

with positive constants M, Γ normalized to unity.

System (1.1)–(1.3) represents respectively the linear momentum balance, the mass balance and a generalized equation for the chemical potential. In a thermodynamical theory due to Gurtin [Gur96] equation (1.3) is identified with a microforce balance. The free energy density underlying system (1.1)–(1.3) has te Landau-Ginzburg-Cahn-Hilliard form

$$f(\varepsilon(\boldsymbol{u}), \chi, \nabla\chi) = W(\varepsilon(\boldsymbol{u}), \chi) + \psi(\chi) + \frac{1}{2}\nabla\chi \cdot \boldsymbol{\Gamma}\nabla\chi, \tag{1.10}$$

with the three terms on the right-hand side representing respectively the elastic, chemical and interfacial energy.

The remaining quantities in (1.1)–(1.3) have the following meaning: $\boldsymbol{b} : \Omega^T \to \mathbb{R}^3$ represents the external body force, and $\boldsymbol{u}_0, \boldsymbol{u}_1 : \Omega \to \mathbb{R}^3$, $\chi_0 : \Omega \to \mathbb{R}$ are the initial conditions respectively for the displacement, the velocity and the order parameter.

The homogeneous boundary conditions in (1.1)–(1.3) are chosen for the sake of simplicity. The condition $(1.1)_3$ means that the body is fixed at the boundary S, $(1.2)_3$ reflects the mass isolation at S, and $(1.3)_2$ is the natural boundary condition for (1.10).

Before discussing the results of the paper let us place our study in the present theory of Cahn-Hilliard systems in elastic solids. In recent years such systems have been the subject of many different modelling, mathematical and numerical studies, we refer e.g. to [MirSchim05a], [MirSchim05b], [BarPaw05], [PawZaj06a] for up to date references. It is known from the materials science literature that the elastic effects have a pronounced effect on the microstructure evolution of the phase separation process and consequently on the resulting material properties.

The most general setting of the Cahn-Hilliard system coupled with elasticity, accounting for additional anisotropic, heterogenous and kinetic effects, was derived by Gurtin [Gur96] within the frame of his thermodynamical theory of phase transitions based on a microforce balance. System (1.1)–(1.3) represents a simplified version of Gurtin's model with neglected anisotropic, heteregeneous and kinetic effects; for more details see [BarPaw05] where the full Gurtin's model was studied.

In view of the fact that the mechanical equilibrium is usually attained on a much faster time scale than diffusion in most of the studies a quasi-stationary approximation of $(1.1)_1$, obtained by neglegting the inertial term $\boldsymbol{u}_{tt}$, was assumed. Various variants of the Cahn-Hilliard system with quasi-stationary elasticity were analyzed by Garcke [Gar00], [Gar03], [Gar05], Bonetti et al. [BCDGSS02], Carrive et al. [CMP00], [CMPR99], Miranville [Mir00], [Mir01a], [Mir01b], [Mir03]. We underline that the results of [Gar00], [Gar03], [BCDGSS02] included mathematically difficult case of nonhomogeneous elasticity with tensor $\boldsymbol{A} = \boldsymbol{A}(\chi)$ depending on the order parameter.

The Cahn-Hilliard system with nonstationary elasticity was studied in [Mir01a], [BarPaw05] where the existence and properites of weak solutions were examined, and in [PawZaj06a] where the classical solvability was proved in 1-D case. It is clear that with the quasi-stationary hypothesis the hyperbolic elasticity system is replaced by the elliptic one and thereby the mathematical analysis becomes qualitatively different.
We point out that the study of the Cahn-Hilliard problem with nonstationary elasticity – apart from the mathematical interest on its own – is of special importance for the initial stages of phase separation at which the formation of the microstructure is on a very fast time scale.

The goal of the present paper is to prove the existence and some regularity of weak solutions to system (1.1)–(1.3). Our ultimate aim is to obtain the existence and

uniqueness of a strong solution to (1.1)–(1.3), i.e. such a solution that its all derivatives appearing in the equations are at least in L_2. The strong solvability theory will be presented separately in [PawZaj06c]. It is based on the regularity results proved in the present paper together with some additional time regularity. More precisely, having sufficiently high time regularity we shall apply the standard elliptic regularity theory to conclude further space regularity and consequently the classical solvability.
As was already mentioned, the strong solvability of system (1.1)–(1.3) in 1-D case was proved in [PawZaj06a]. The strong solvability of the single Cahn-Hilliard equation in 1-D and 3-D cases was analyzed firstly by Elliott and Zheng [ElZh86].

We point out that in three space dimensions the coupled system (1.1)–(1.3) shows features that make its analysis much more difficult than in one-dimensional setting.
The arguments used by the authors in the single space dimension in [PawZaj06a], based on the space regularity of the wave equation, do not extend to the 3-D case.

The key idea of the regularity theory presented here and in the forthcoming paper [PawZaj06c] consists in the analysis of time-differentiated versions of problem (1.1)–(1.3) which yield solutions with sufficiently high time derivatives. The analysis is performed with the help of the Faedo-Galerkin approximation. The procedure is straightforward except of some difficulties of technical nature due to many nonlinear terms that appear in the system after differentiation with respect to time variable.

For further analysis it is convenient to introduce a simplified formulation of problem (1.1)–(1.3) which results on account of particular constitutive equations (1.4)–(1.6) and (1.9). Let $\boldsymbol{Q}$ be the linear elasticity operator defined by

$$\boldsymbol{u} \mapsto \boldsymbol{Q}\boldsymbol{u} = \nabla \cdot \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \bar{\mu}\Delta \boldsymbol{u} + (\bar{\lambda} + \bar{\mu})\nabla(\nabla \cdot \boldsymbol{u}). \tag{1.11}$$

Moreover, let us denote

$$\boldsymbol{B} = -\boldsymbol{A}(\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad D = -\boldsymbol{B} \cdot (\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad E = -\boldsymbol{B} \cdot \bar{\boldsymbol{\varepsilon}}_a, \tag{1.12}$$

where $B = (B_{ij})$ is a symmetric second order tensor, and D, E are two scalars. With such notation we have

$$\begin{aligned} \nabla \cdot W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) &= \nabla \cdot \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{A}(\bar{\boldsymbol{\varepsilon}}_a + z(\chi)(\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a)) \\ &= \boldsymbol{Q}\boldsymbol{u} + z'(\chi)\boldsymbol{B}\nabla\chi, \end{aligned} \tag{1.13}$$

and

$$W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = z'(\chi)(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E),$$

so that (1.1)–(1.3) simplifies to

$$\begin{aligned} &\boldsymbol{u}_{tt} - \boldsymbol{Q}\boldsymbol{u} = z'(\chi)\boldsymbol{B}\nabla\chi + \boldsymbol{b} && \text{in } \Omega^T, \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega, \\ &\boldsymbol{u} = 0 && \text{on } S^T, \end{aligned} \tag{1.14}$$

$$
(1.15) \qquad \begin{aligned} &\chi_t - \Delta\mu = 0 && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla\mu = 0 && \text{on } S^T, \end{aligned}
$$

$$
(1.16) \qquad \begin{aligned} &\mu = -\Delta\chi + \psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi) && \text{in } \Omega^T, \\ &\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on } S^T, \end{aligned}
$$

with $W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi)$ given by $(1.13)_2$.

Let us note that the combined systems (1.15) and (1.16) yield the following Cahn--Hilliard problem

$$
(1.17) \qquad \begin{aligned} &\chi_t + \Delta^2\chi = \Delta[\psi'(\chi) + z'(\chi)(\boldsymbol{B}\cdot\varepsilon(\boldsymbol{u}) + Dz(\chi) + E)] && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on } S^T, \\ &\boldsymbol{n}\cdot\nabla\Delta\chi = z'(\chi)\boldsymbol{n}\cdot\nabla(\boldsymbol{B}\cdot\varepsilon(\boldsymbol{u})) && \text{on } S^T, \end{aligned}
$$

coupled with the elasticity system (1.14). It is seen that the problems are coupled not only through the right-hand sides but also through the boundary conditions. Moreover, by definition (1.7) of the shape function z, the problems uncouple for $\chi \le -1$ and $\chi \ge 1$. We point out that the boundary coupling is characteristic for the multidimensional problem and does not appear in its one-dimensional setting. In fact, in 1-D case assuming that $b = 0$ on S^T, it follows from $(1.14)_1$, $(1.14)_3$ and $(1.17)_3$ that $u_{xx} = 0$ on S^T, and consequently condition $(1.17)_4$ yields $\chi_{xxx} = 0$ on S^T. This fact was used in [PawZaj06a] in the analysis of the 1-D version of problem (1.1)–(1.3).

The plan of the paper is as follows: In Section 2 we present our main assumptions and results stated in Theorems 2.1 and 2.2. Theorem 2.1 asserts the existence of a weak solution to problem (1.1)–(1.3). Theorem 2.2 provides a time-regularity result obtained by differentiating (1.1)–(1.3) with respect to time variable. In Section 3 we introduce a Faedo-Galerkin approximation of (1.1)–(1.3). We derive primary energy estimates with constants independent of approximation and time, and investigate their implications. In Section 4 we consider a time-differentiated version of the approximate problem and establish the first regularity estimates with constants uniform in approximation but depending on time. The subsequent sections 5 and 6 provide the existence proofs respectively of Theorems 2.1 and 2.2. The proofs are based on the previously established uniform a priori estimates which, by standard arguments, allow to pass to the limit in the corresponding versions of the approximate problems.

We remark that having in mind a future examination of the long time behaviour of solutions we record time-dependences of various constants. The obtained regularity estimates turn out to depend exponentially on time, thus in the present form are not useful for the long time analysis. We point out that in long-time analysis the crucial point is to show property that if $\chi(0) \in [-1,1]$ then $\chi(t) \in [-1,1]$ for all $t > 0$, in other words that the order parameter attains physically meaningful values for all times. This question is left open in the present paper.

We use the following notation:

$$\begin{aligned}
&\boldsymbol{x} = (x_i)_{i=1,2,3} \in \Omega \subset \mathbb{R}^3 \quad \text{the material point,}\\
&f_{,i} = \frac{\partial f}{\partial x_i}, \quad f_t = \frac{df}{dt} \quad \text{the material space and time derivatives,}\\
&\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1,2,3}, \quad W_{,\varepsilon}(\boldsymbol{\varepsilon},\chi) = \left(\frac{\partial W(\boldsymbol{\varepsilon},\chi)}{\partial \varepsilon_{ij}}\right)_{i,j=1,2,3},\\
&W_{,\chi}(\boldsymbol{\varepsilon},\chi) = \frac{\partial W(\boldsymbol{\varepsilon},\chi)}{\partial \chi}, \quad \psi'(\chi) = \frac{d\psi(\chi)}{d\chi}.
\end{aligned}$$

For simplicity, whenever there is no danger of confusion, we omit the arguments $(\boldsymbol{\varepsilon},\chi)$. The specification of tensor indices is omitted as well.

Vector- and tensor-valued mappings are denoted by bold letters.

The summation convention over repeated indices is used, as well as the notation: for vectors $\boldsymbol{a} = (a_i)$, $\tilde{\boldsymbol{a}} = (a_i)$ and tensors $\boldsymbol{B} = (B_{ij})$, $\tilde{\boldsymbol{B}} = (\tilde{B}_{ij})$, $\boldsymbol{A} = (A_{ijkl})$, we write

$$\begin{aligned}
&\boldsymbol{a}\cdot\tilde{\boldsymbol{a}} = a_i\tilde{a}_i, && \boldsymbol{B}\cdot\tilde{\boldsymbol{B}} = B_{ij}\tilde{B}_{ij},\\
&\boldsymbol{AB} = (A_{ijkl}B_{kl}), && \boldsymbol{BA} = (B_{ij}A_{ijkl}),\\
&|\boldsymbol{a}| = (a_ia_i)^{1/2}, && |\boldsymbol{B}| = (B_{ij}B_{ij})^{1/2}.
\end{aligned}$$

The symbols ∇ and $\nabla\cdot$ denote the gradient and the divergence operators with respect to the material point $\boldsymbol{x}$. For the divergence of a tensor field we use the convention of the contraction over the last index, e.g. $\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{x}) = (\varepsilon_{ij,j}(\boldsymbol{x}))$.

We use the standard Sobolev spaces notation $H^m(\Omega) = W_2^m(\Omega)$ for $m \in \mathbb{N}$. Besides,

$$\begin{aligned}
H_0^1(\Omega) &= \{v \in H^1(\Omega):\ v = 0 \text{ on } S\},\\
H_N^2(\Omega) &= \{v \in H^2(\Omega):\ \boldsymbol{n}\cdot\nabla v = 0 \text{ on } S\},
\end{aligned}$$

where $\boldsymbol{n}$ is the outward unit normal to $S = \partial\Omega$, denote the subspaces respectively of $H^1(\Omega)$ and $H^2(\Omega)$, with the standard norms of $H^1(\Omega)$ and $H^2(\Omega)$.

By bold letters we denote the spaces of vector or tensor-valued functions, e.g.

$$\boldsymbol{L}_2(\Omega) = (L_2(\Omega))^n, \quad \boldsymbol{H}^1(\Omega) = (H^1(\Omega))^n, \quad n \in \mathbb{N};$$

if there is no confusion we do not specify dimension n.

Moreover, we write

$$\|\boldsymbol{a}\|_{L_2(\Omega)} = \|\,|\boldsymbol{a}|\,\|_{L_2(\Omega)}, \quad \|\boldsymbol{a}\|_{H^1(\Omega)} = \|\,|\boldsymbol{a}|\,\|_{L_2(\Omega)} + \|\,|\nabla\boldsymbol{a}|\,\|_{L_2(\Omega)}$$

for the corresponding norms of a vector-valued function $\boldsymbol{a}(\boldsymbol{x}) = (a_i(\boldsymbol{x}))$; similarly for tensor-valued functions.

As common, the symbol $(\cdot,\cdot)$ denotes the scalar product in $L_2(\Omega)$. For simplicity, we use the same symbol to denote scalar products in $\boldsymbol{L}_2(\Omega) = (L_2(\Omega))^n$, e.g. we write

$$(a,\tilde{a}) = \int_\Omega a(\boldsymbol{x})\tilde{a}(\boldsymbol{x})dx, \quad (\boldsymbol{a},\tilde{\boldsymbol{a}}) = \int_\Omega a_i(\boldsymbol{x})\tilde{a}_i(\boldsymbol{x})dx,$$

$$(\boldsymbol{B},\tilde{\boldsymbol{B}}) = \int_\Omega B_{ij}(\boldsymbol{x})\tilde{B}_{ij}(\boldsymbol{x})dx.$$

The dual of the space V is denoted by V', and $\langle\cdot,\cdot\rangle_{V',V}$ stands for the duality pairing between V' and V.
By c and $c(T)$ we denote generic positive constants different in various instances, depending on the data of the problem and domain Ω; whenever it is of interest their dependence on parameters is specified. The argument T indicates the time horizon dependence. Moreover, δ denotes a generic, sufficiently small positive constant.

2. Assumptions and main results

System (1.1)–(1.3) (in simplified form (1.14)–(1.16)) is studied under the following assumptions:

(A1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary S of class at least C^2; $T > 0$ is an arbitrary final time.

(A2) The coefficients of the elasticity operator defined by (1.11) $\boldsymbol{Q}$ satisfy

$$\bar{\mu} > 0, \quad 3\bar{\lambda} + 2\bar{\mu} > 0 \ \text{(elasticity range)}. \tag{2.1}$$

These two conditions assure the following:

(i) Coercivity and boundedness of the operator $\boldsymbol{A}$

$$\underline{c}|\boldsymbol{\varepsilon}|^2 \le \boldsymbol{\varepsilon}\cdot\boldsymbol{A}\boldsymbol{\varepsilon} \le \bar{c}|\boldsymbol{\varepsilon}|^2 \quad \text{for all} \ \ \boldsymbol{\varepsilon} \in \boldsymbol{S}^2, \tag{2.2}$$

where $\boldsymbol{S}^2$ denotes the set of symmetric second order tensors in $\mathbb{R}^3$, and

$$\underline{c} = \min\{3\bar{\lambda} + 2\bar{\mu}, 2\bar{\mu}\}, \quad \bar{c} = \max\{3\bar{\lambda} + 2\bar{\mu}, 2\bar{\mu}\};$$

(ii) Strong ellipticity of the operator $\boldsymbol{Q}$ (property holding true under weaker assumption $\bar{\mu} > 0$, $\bar{\lambda}+2\bar{\mu} > 0$, see [PawZoch02], Section 7). Thanks to this property the following estimate holds true (see [Nec67], Lemma 3.2):

$$c\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \le \|\boldsymbol{Q}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega)} \quad \text{for} \quad \boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega) \tag{2.3}$$

with constant c depending on Ω.

Hence, since clearly $\|\boldsymbol{Q}\boldsymbol{u}\|_{L_2(\Omega)} \le \bar{\bar{c}}\|\boldsymbol{u}\|_{H^2(\Omega)}$, it follows that the norms $\|\boldsymbol{Q}\boldsymbol{u}\|_{L_2(\Omega)}$ and $\|\boldsymbol{u}\|_{H^2(\Omega)}$ are equivalent on $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$.

The next two assumptions concern the ingredients of the free energy (see (1.10) with $\boldsymbol{\Gamma} = \boldsymbol{I}$)

$$f(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi, \nabla\chi) = W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) + \psi(\chi) + \frac{1}{2}|\nabla\chi|^2. \tag{2.4}$$

(A3) The elastic energy $W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)$ is given by (1.4)–(1.6). The interpolation function $z : \mathbb{R} \to [0,1]$ in definition (1.6) of $\bar{\boldsymbol{\varepsilon}}(\chi)$ is at least of class C^1 with the property (1.7). Hence,

$$0 \le z(\chi) \le 1 \quad \text{and} \quad |z'(\chi)| \le c \quad \text{for all} \quad \chi \in \mathbb{R}. \tag{2.5}$$

(A4) The chemical energy $\psi(\chi)$ has the form of the standard double-well potential (1.8), so

$$\psi'(\chi) = \chi^3 - \chi, \quad \psi''(\chi) = 3\chi^2 - 1, \quad \psi'''(\chi) = 6\chi. \tag{2.6}$$

Moreover, for simplicity it is assumed that

(A5) The mobility tensor $\boldsymbol{M}$ and the interfacial tensor $\boldsymbol{\Gamma}$ are the identities matrices $\boldsymbol{M} = \boldsymbol{I}$, $\boldsymbol{\Gamma} = \boldsymbol{I}$.

The second order symmetric tensor $\boldsymbol{B}$ and the scalars D, E are defined in (1.12).

We note that assumptions (A3) and (A4) imply the following bounds for all $\boldsymbol{\varepsilon} \in \boldsymbol{S}^2$ and $\chi \in \mathbb{R}$:

$$\begin{aligned}
&|\bar{\boldsymbol{\varepsilon}}(\chi)| \le |\bar{\boldsymbol{\varepsilon}}_a| + |\bar{\boldsymbol{\varepsilon}}_b| \le c,\\
&|\bar{\boldsymbol{\varepsilon}}'(\chi)| = |z'(\chi)(\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a)| \le c,\\
&|W(\boldsymbol{\varepsilon}, \chi)| \le \frac{1}{2}\bar{c}|\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}(\chi)|^2 \le c(|\boldsymbol{\varepsilon}|^2 + 1),\\
&|W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \chi)| + |W_{,\chi}(\boldsymbol{\varepsilon}, \chi)| \le c(|\boldsymbol{\varepsilon}| + 1),\\
&|\psi(\chi)| \le c(\chi^4 + 1), \quad |\psi'(\chi)| \le c(|\chi|^3 + 1)
\end{aligned} \tag{2.7}$$

with some positive constant c. Moreover, by the Young inequality, we have

$$W(\boldsymbol{\varepsilon}, \chi) \ge \frac{1}{2}\underline{c}|\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}(\chi)|^2 \ge \frac{1}{4}\underline{c}|\boldsymbol{\varepsilon}|^2 - \frac{1}{2}\underline{c}|\bar{\boldsymbol{\varepsilon}}(\chi)|^2 \ge \frac{1}{4}\underline{c}|\boldsymbol{\varepsilon}|^2 - \underline{c}(|\bar{\boldsymbol{\varepsilon}}_a|^2 + |\bar{\boldsymbol{\varepsilon}}_b|^2), \tag{2.8}$$

and

$$\psi(\chi) \ge \frac{1}{8}\chi^4 - \frac{1}{4}.$$

This shows that free energy (2.4) satisfies the following structure condition

$$\begin{aligned}
f(\boldsymbol{\varepsilon}, \chi, \nabla\chi) &\ge \frac{1}{4}\underline{c}|\boldsymbol{\varepsilon}|^2 + \frac{1}{8}\chi^4 + \frac{1}{2}|\nabla\chi|^2 - \underline{c}(|\bar{\boldsymbol{\varepsilon}}_a|^2 + |\bar{\boldsymbol{\varepsilon}}_b|^2) - \frac{1}{4}\\
&\ge c_f(|\boldsymbol{\varepsilon}|^2 + \chi^4 + |\nabla\chi|^2) - c_f'
\end{aligned} \tag{2.9}$$

with constants $c_f > 0$ and c'_f given by

$$c_f = \min\left\{\frac{1}{4}\underline{c}, \frac{1}{8}\right\}, \quad c'_f = \underline{c}(|\bar{\varepsilon}_a|^2 + |\bar{\varepsilon}_b|^2) - \frac{1}{4}.$$

This bound plays the key role in the derivation of energy estimates for problem (1.1)-(1.3) (see Section 4).

For further purposes we recall here the following two additional properties of the operator $\boldsymbol{Q}$:

(2.10) $\qquad \boldsymbol{Q}$ is selfadjoint on $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$, i.e.

$$\begin{aligned}(\boldsymbol{Q}\boldsymbol{u}, \boldsymbol{v}) &= -\bar{\mu}(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\bar{\lambda} + \bar{\mu})(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) \\ &= (\boldsymbol{u}, \boldsymbol{Q}\boldsymbol{v}) \quad \text{for} \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega),\end{aligned}$$

(2.11) $\qquad -\boldsymbol{Q}$ is positive on $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$, i.e.

$$\begin{aligned}(-\boldsymbol{Q}\boldsymbol{u}, \boldsymbol{u}) = \bar{\mu}\|\nabla \boldsymbol{u}\|^2_{L_2(\Omega)} + (\bar{\lambda} + \bar{\mu})\|\nabla \cdot \boldsymbol{u}\|^2_{L_2(\Omega)} \geq 0 \\ \text{for} \quad \boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega).\end{aligned}$$

We state now the main results of the paper.

Theorem 2.1. *Weak solutions*
Let assumptions (A1)-(A5) hold true. Moreover, let the data satisfy

$$\begin{aligned}&\boldsymbol{b} \in \boldsymbol{L}_2(\Omega^T), \\ &\boldsymbol{u}_0 \in \boldsymbol{H}_0^1(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{L}_2(\Omega), \quad \chi_0 \in H^1(\Omega).\end{aligned} \tag{2.12}$$

Then there exist functions $(\boldsymbol{u}, \chi, \mu)$ *such that*

$$\begin{aligned}&\boldsymbol{u} \in L_\infty(0,T; \boldsymbol{H}_0^1(\Omega)), \quad \boldsymbol{u}_t \in L_\infty(0,T; \boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_{tt} \in L_2(0,T; (\boldsymbol{H}_0^1(\Omega))'), \\ &\chi \in L_\infty(0,T; H^1(\Omega)) \cap L_2(0,T; H_N^2(\Omega)), \quad \chi_t \in L_2(0,T; (H^1(\Omega))'), \\ &\mu \in L_2(0,T; H^1(\Omega)), \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \chi(0) = \chi_0,\end{aligned} \tag{2.13}$$

which satisfy problem (1.14)–(1.16) in the following weak sense

$$(2.14)\qquad \begin{aligned}
&\int_0^T \langle \boldsymbol{u}_{tt}, \boldsymbol{\eta}\rangle_{(\boldsymbol{H}_0^1(\Omega))', \boldsymbol{H}_0^1(\Omega)} dt + \int_0^T (A\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))dt \\
&\quad = \int_0^T (z'(\chi)B\nabla\chi + \boldsymbol{b}, \boldsymbol{\eta})dt \quad \forall \boldsymbol{\eta} \in L_2(0,T;\boldsymbol{H}_0^1(\Omega)), \\
&\int_0^T \langle \chi_t, \xi\rangle_{(H^1(\Omega))', H^1(\Omega)} dt + \int_0^T (\nabla\mu, \nabla\xi)dt = 0 \\
&\quad \forall \xi \in L_2(0,T;H^1(\Omega)), \\
&\int_0^T (\mu, \zeta)dt = -\int_0^T (\Delta\chi, \zeta)dt + \int_0^T (\psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi), \zeta)dt \\
&\quad \forall \zeta \in L_2(0,T;L_2(\Omega)).
\end{aligned}$$

Moreover, $(\boldsymbol{u}, \chi, \mu)$ satisfy a priori estimates

$$(2.15)\qquad \begin{aligned}
&\|\boldsymbol{u}_t\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\chi\|_{L_\infty(0,T;L_4(\Omega))} \\
&\quad + \|\nabla\chi\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\nabla\mu\|_{\boldsymbol{L}_2(\Omega^T)} + \|\chi_t\|_{L_2(0,T;(H^1(\Omega))')} \le c_0, \\
&\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{H}_0^1(\Omega))} + \|\chi\|_{L_\infty(0,T;H^1(\Omega))} \le c_1, \\
&\|\chi\|_{L_2(0,T;H_N^2(\Omega))} + \|\mu\|_{L_2(0,T;H^1(\Omega))} \le c_2(T), \\
&\|\boldsymbol{u}_{tt}\|_{L_2(0,T;(\boldsymbol{H}_0^1(\Omega))')} \le c_3(T),
\end{aligned}$$

with positive constants

$$\begin{aligned}
&c_0 = c(\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}, \|\boldsymbol{u}_1\|_{\boldsymbol{L}_2(\Omega)}, \|\chi_0\|_{H^1(\Omega)}, \|\boldsymbol{b}\|_{L_1(0,T;\boldsymbol{L}_2(\Omega))}, c_f, c_f'), \\
&c_1 = c(c_0, \Omega), \quad c_2(T) = c(c_1)T^{1/2}, \quad c_3(T) = c(c_0, \|\boldsymbol{b}\|_{\boldsymbol{L}_2(\Omega^T)})T^{1/2}.
\end{aligned}$$

The second theorem states a time-regularity result which is concluded from a time--differentiated version of problem (1.14)–(1.16).

In compatibility with equations $(1.14)_1$, $(1.15)_1$ and $(1.16)_1$, we define the following initial conditions corresponding respectively to $\boldsymbol{u}_{tt}(0)$, $\chi_t(0)$ and $\mu(0)$:

$$(2.16)\qquad \begin{aligned}
&\boldsymbol{u}_2 := \boldsymbol{Q}\boldsymbol{u}_0 + z'(\chi_0)B\nabla\chi_0 + \boldsymbol{b}(0), \\
&\chi_1 := \Delta\mu_0, \\
&\mu_0 := -\Delta\chi_0 + \psi'(\chi_0) + z'(\chi_0)(B \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_0) + Dz(\chi_0) + E).
\end{aligned}$$

Theorem 2.2. *Time regularity*
Let (A1)–(A5) hold, the boundary S of domain Ω be of class C^4, and

$$z : \mathbb{R} \to [0,1] \text{ be of class } C^2 \text{ with} \quad (2.17)$$
$$|z'(\chi)| + |z''(\chi)| \le c \text{ for all } \chi \in \mathbb{R}.$$

Morever, let the data satisfy

$$\begin{aligned}
&\boldsymbol{b} \in H^1(0,T; \boldsymbol{L}_2(\Omega)),\\
(2.18)\qquad &\boldsymbol{u}_0 \in \boldsymbol{H}^3(\Omega) \cap \boldsymbol{H}_0^1(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{H}_0^1(\Omega), \quad \chi_0 \in H^4(\Omega) \cap H_N^2(\Omega),\\
&\boldsymbol{u}_2 \in \boldsymbol{L}_2(\Omega), \quad \chi_1 \in L_2(\Omega), \quad \mu_0 \in H_N^2(\Omega).
\end{aligned}$$

Then there exist functions $(\boldsymbol{u}, \chi, \mu)$ such that

$$\begin{aligned}
&\boldsymbol{u} \in L_\infty(0,T; \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)), \quad \boldsymbol{u}_t \in L_\infty(0,T; \boldsymbol{H}_0^1(\Omega)),\\
&\boldsymbol{u}_{tt} \in L_\infty(0,T; \boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_{ttt} \in L_2(0,T; (\boldsymbol{H}_0^1(\Omega))'),\\
&\chi \in C^{1/2}([0,T]; H_N^2(\Omega)), \quad \chi_t \in L_\infty(0,T; L_2(\Omega)) \cap L_2(0,T; H_N^2(\Omega)),\\
(2.19)\qquad &\chi_{tt} \in L_2(0,T; (H_N^2(\Omega))'),\\
&\mu \in L_\infty(0,T; H_N^2(\Omega)), \quad \mu_t \in L_2(\Omega^T),\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \boldsymbol{u}_{tt}(0) = \boldsymbol{u}_2,\\
&\chi(0) = \chi_0, \quad \chi_t(0) = \chi_1, \quad \mu(0) = \mu_0,
\end{aligned}$$

which satisfy problem (1.14)–(1.16) in the sense of the identities:

$$\begin{aligned}
&\int_0^T \langle \boldsymbol{u}_{ttt}, \boldsymbol{\eta} \rangle_{(\boldsymbol{H}_0^1(\Omega))', \boldsymbol{H}_0^1(\Omega)} dt + \int_0^T (\boldsymbol{A}\varepsilon(\boldsymbol{u}_t), \varepsilon(\boldsymbol{\eta})) dt\\
&\quad = \int_0^T ([z'(\chi)\boldsymbol{B}\nabla\chi]_{,t} + \boldsymbol{b}_t, \boldsymbol{\eta}) dt \quad \forall \boldsymbol{\eta} \in L_2(0,T; \boldsymbol{H}_0^1(\Omega)),\\
(2.20)\qquad &\int_0^T \langle \chi_{tt}, \xi \rangle_{(H_N^2(\Omega))', H_N^2(\Omega)} dt = \int_0^T (\mu_t, \Delta\xi) dt\\
&\qquad \forall \xi \in L_2(0,T; H_N^2(\Omega)),\\
&\int_0^T (\mu_t, \zeta) dt = -\int_0^T (\Delta\chi_t, \zeta) dt + \int_0^T ([\psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}), \chi)]_{,t}, \zeta) dt\\
&\qquad \forall \zeta \in L_2(0,T; L_2(\Omega)),
\end{aligned}$$

where

$$\begin{aligned}
&[z'(\chi)\boldsymbol{B}\nabla\chi]_{,t} = z''(\chi)\chi_t\boldsymbol{B}\nabla\chi + z'(\chi)\boldsymbol{B}\nabla\chi_t,\\
(2.21)\qquad &[\psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}), \chi)]_{,t} = \psi''(\chi)\chi_t + z''(\chi)\chi_t(\boldsymbol{B} \cdot \varepsilon(\boldsymbol{u}) + Dz(\chi) + E)\\
&\qquad + z'(\chi)(\boldsymbol{B} \cdot \varepsilon(\boldsymbol{u}_t) + Dz'(\chi)\chi_t).
\end{aligned}$$

In addition, $(\boldsymbol{u},\chi,\mu)$ satisfy estimates (2.15) and

$$
(2.22)\quad
\begin{aligned}
&\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{H}^2(\Omega))}+\|\boldsymbol{u}_t\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))}+\|\boldsymbol{u}_{tt}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))}\le c_5(T),\\
&\|\chi\|_{C^{1/2}([0,T];H^2_N(\Omega))}+\|\chi_t\|_{L_\infty(0,T;L_2(\Omega))}+\|\chi_t\|_{L_2(0,T;H^2_N(\Omega))}\\
&\quad+\|\mu\|_{L_\infty(0,T;H^2_N(\Omega))}\le c_4(T),\\
&\|\boldsymbol{u}_{ttt}\|_{L_2(0,T;(\boldsymbol{H}^1_0(\Omega))')}+\|\chi_{tt}\|_{L_2(0,T;(H^2_N(\Omega))')}+\|\mu_t\|_{L_2(\Omega^T)}\le T^{1/2}c_5(T),
\end{aligned}
$$

with constants

$$
\begin{aligned}
&c_4(T)=c(T^{1/2}E_1(T)+\|\chi_1\|_{L_2(\Omega)})[\exp a(T)]^{1/2},\\
&c_5(T)=T^{1/2}c_4(T),
\end{aligned}
$$

where

$$
\begin{aligned}
&E_1(T)=T^{1/2}\|\boldsymbol{b}_t\|_{L_2(\Omega^T)}+\|\boldsymbol{u}_2\|_{L_2(\Omega)}+\|\varepsilon(\boldsymbol{u}_1)\|_{L_2(\Omega)},\\
&a(T)=c(c_0)T^8\exp(cT).
\end{aligned}
$$

3. The Faedo-Galerkin approximation

In this section we introduce a Faedo-Galerkin approximation of problem (1.1)–(1.3) (in simplified form (1.14)–(1.16)) and derive basic energy estimates. These estimates are used to prove the existence of weak solutions in Theorem 2.1. Throughout this section we assume that the domain Ω has the boundary S at least of class C^2.

3.1. Approximation

Let us consider the following two eigenvalue problems

$$
(3.1)\quad
\begin{aligned}
-\boldsymbol{Q}\boldsymbol{v}_j&=\lambda_j\boldsymbol{v}_j &&\text{in }\ \Omega,\\
\boldsymbol{v}_j&=\boldsymbol{0} &&\text{on }\ S,\quad j\in\mathbb{N},
\end{aligned}
$$

where $\boldsymbol{Q}$ is the elliptic operator defined by (1.11), and

$$
(3.2)\quad
\begin{aligned}
-\Delta w_j&=\lambda_j w_j &&\text{in }\ \Omega,\\
\boldsymbol{n}\cdot\nabla w_j&=0 &&\text{on }\ S,\quad j\in\mathbb{N}.
\end{aligned}
$$

We recall that, by virtue of the elliptic regularity theory, if the domain Ω has the boundary of class C^l, $l\in\mathbb{N}$, then the solutions of (3.1) and (3.2) satisfy

$$
\boldsymbol{v}_j\in\boldsymbol{H}^l(\Omega),\quad w_j\in H^l(\Omega).
$$

We take the family $\{\boldsymbol{v}_j\}_{j\in\mathbb{N}}$ as a basis of the space $\boldsymbol{H}_0^1(\Omega)$ and the family $\{w_j\}_{j\in\mathbb{N}}$ as a basis of the space

$$H_N^2(\Omega) = \{w \in H^2(\Omega) : \boldsymbol{n}\cdot\nabla w = 0 \text{ on } S\}.$$

Such choice is possible thanks to the following properties of (3.1) and (3.2). On account of (2.10) we have

$$\begin{aligned}\lambda_i(\boldsymbol{v}_i,\boldsymbol{v}_j) &= (-\boldsymbol{Q}\boldsymbol{v}_i,\boldsymbol{v}_j)\\ &= \bar{\mu}(\nabla\boldsymbol{v}_i,\nabla\boldsymbol{v}_j) + (\bar{\lambda}+\bar{\mu})(\nabla\cdot\boldsymbol{v}_i,\nabla\cdot\boldsymbol{v}_j)\\ &= (\boldsymbol{v}_i,-\boldsymbol{Q}\boldsymbol{v}_j) = \lambda_j(\boldsymbol{v}_i,\boldsymbol{v}_j) \quad \text{for } \ i,j\in\mathbb{N}.\end{aligned} \tag{3.3}$$

Identities (3.3) show, by the Poincaré-Friedrichs inequality, that the family $\{\boldsymbol{v}_j\}_{j\in\mathbb{N}}$ is orthogonal in $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{L}_2(\Omega)$ scalar products.
We shall assume that $\boldsymbol{v}_j$ are normalized so that $(\boldsymbol{v}_i,\boldsymbol{v}_i)=1$. Thereby the basis $\{\boldsymbol{v}_j\}_{j\in\mathbb{N}}$ becomes orthonormal in $\boldsymbol{L}_2(\Omega)$ and orthogonal in $\boldsymbol{H}^1(\Omega)$ scalar products.

Similarly, the family $\{w_j\}_{j\in\mathbb{N}}$ satisfies

$$\begin{aligned}\lambda_i(w_i,w_j) &= (-\Delta w_i,w_j) = (\nabla w_i,\nabla w_j)\\ &= (w_i,-\Delta w_j) = \lambda_j(w_i,w_j),\end{aligned} \tag{3.4}$$

and

$$\lambda_i\lambda_j(w_i,w_j) = (\Delta w_i,\Delta w_j) \quad \text{for } \ i,j\in\mathbb{N}.$$

Hence, by the Poincaré inequality, it follows that the family $\{w_j\}_{j\in\mathbb{N}}$ is orthogonal in $H^2(\Omega)$, $H^1(\Omega)$ and $L_2(\Omega)$ scalar products.
We normalize w_j so that $(w_i,w_i)=1$. Then the basis $\{w_j\}_{j\in\mathbb{N}}$ becomes orthonormal in $L_2(\Omega)$ and orthogonal in $H^1(\Omega)$ and $H^2(\Omega)$ scalar products.
Furthermore, we assume without loss of generality that $w_1 = 1$.

For $m\in\mathbb{N}$ we denote by

$$\boldsymbol{V}_{0m} = \operatorname{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\} \quad \text{and} \quad V_m = \operatorname{span}\{w_1,\dots,w_m\}$$

the finite dimensional subspaces, respectively of $\boldsymbol{H}_0^1(\Omega)$ and $H_N^2(\Omega)$, spanned by $\{\boldsymbol{v}_1,\dots$ $\dots,\boldsymbol{v}_m\}$ and $\{w_1,\dots,w_m\}$.

Now, let us introduce the following approximation of problem (1.14)–(1.16): For any $m\in\mathbb{N}$ find a triple of functions $(\boldsymbol{u}^m,\chi^m,\mu^m)$ of the form

$$\begin{aligned}&\boldsymbol{u}^m(\boldsymbol{x},t) = \sum_{i=1}^m e_i^m(t)\boldsymbol{v}_i(\boldsymbol{x}), \quad \chi^m(\boldsymbol{x},t) = \sum_{i=1}^m c_i^m(t)w_i(\boldsymbol{x}),\\ &\mu^m(\boldsymbol{x},t) = \sum_{i=1}^m d_i^m(t)w_i(\boldsymbol{x}),\end{aligned} \tag{3.5}$$

with $e_i^m(t)$, $c_i^m(t)$, $d_i^m(t)$ being determined so that

$$
\begin{aligned}
&(\boldsymbol{u}_{tt}^m, \boldsymbol{v}_j) + (A\varepsilon(\boldsymbol{u}^m), \varepsilon(\boldsymbol{v}_j)) = (z'(\chi^m)B\nabla\chi^m + \boldsymbol{b}, \boldsymbol{v}_j),\\
&(\chi_t^m, w_j) + (\nabla\mu^m, \nabla w_j) = 0,\\
&(\mu^m, w_j) = -(\Delta\chi^m, w_j) + (\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m), w_j), \quad j = 1, \ldots, m,\\
&\boldsymbol{u}^m(0) = \boldsymbol{u}_0^m, \quad \boldsymbol{u}_t^m(0) = \boldsymbol{u}_1^m, \quad \chi^m(0) = \chi_0^m,
\end{aligned}
\tag{3.6}
$$

where $\boldsymbol{u}_0^m, \boldsymbol{u}_1^m \in \boldsymbol{V}_{0m}$ and $\chi_0^m \in V_m$ are the projections respectively of $\boldsymbol{u}_0, \boldsymbol{u}_1$ and χ_0 satisfying for $m \to \infty$

$$
\begin{aligned}
&\boldsymbol{u}_0^m \to \boldsymbol{u}_0 \ \text{ strongly in } \boldsymbol{H}_0^1(\Omega),\\
&\boldsymbol{u}_1^m \to \boldsymbol{u}_1 \ \text{ strongly in } \boldsymbol{L}_2(\Omega),\\
&\chi_0^m \to \chi_0 \ \text{ strongly in } H^1(\Omega).
\end{aligned}
\tag{3.7}
$$

Clearly, (3.6) can be expressed as a system of first order ordinary differential equations for the coefficients $(e_1^m, \ldots, e_m^m)$, $(e_{1,t}^m, \ldots, e_{m,t}^m)$, $(c_1^m, \ldots, c_m^m)$, with the right-hand sides being by assumptions continuous functions of their arguments. Thus (3.6) has a solution local in time on an interval $[0, T_m]$, $T_m > 0$. The uniform in m a priori estimates proved in lemmas below show that $T_m = T$, i.e. (3.6) has a solution on the interval $[0, T]$.

3.2. Energy estimates

Lemma 3.1. *Let (A1)–(A5) hold and the data satisfy*

$$
\boldsymbol{u}_0 \in \boldsymbol{H}_0^1(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{L}_2(\Omega), \quad \chi_0 \in H^1(\Omega), \quad \boldsymbol{b} \in L_1(0, T; \boldsymbol{L}_2(\Omega)).
\tag{3.8}
$$

Then a solution $(\boldsymbol{u}^m, \chi^m, \mu^m)$ to problem (3.6) satisfies the following uniform estimate

$$
\begin{aligned}
&\|\boldsymbol{u}_t^m\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\varepsilon(\boldsymbol{u}^m)\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\chi^m\|_{L_\infty(0,T;L_4(\Omega))}\\
&\quad + \|\nabla\chi^m\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\nabla\mu^m\|_{\boldsymbol{L}_2(\Omega^T)} \le c_0,
\end{aligned}
\tag{3.9}
$$

with the constant

$$
c_0 = c_0(\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}, \|\boldsymbol{u}_1\|_{\boldsymbol{L}_2(\Omega)}, \|\chi_0\|_{H^1(\Omega)}, \|\boldsymbol{b}\|_{L_1(0,T;\boldsymbol{L}_2(\Omega))}, c_f, c_f').
$$

Proof. We derive energy identity for system (3.6). Firstly, let us note that, according to (1.13),

$$
z'(\chi)B\nabla\chi = -\nabla \cdot A(\varepsilon_a + z(\chi)(\bar{\varepsilon}_b - \bar{\varepsilon}_a)),
$$

thus an equivalent form of $(3.6)_1$ is

$$
(\boldsymbol{u}_{tt}^m, \boldsymbol{v}_j) + (W_{,\varepsilon}(\varepsilon(\boldsymbol{u}^m), \chi^m), \varepsilon(\boldsymbol{v}_j)) = (\boldsymbol{b}, \boldsymbol{v}_j).
\tag{3.10}
$$

Testing (3.10) by $\boldsymbol{u}_t^m(t)$ (i.e. multiplying by $e_j^m(t)$ and summing over j from $j = 1$ to $j = m$) gives

$$(3.11) \qquad \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t^m\|^2_{L_2(\Omega)} + (W_{,\varepsilon}(\varepsilon(\boldsymbol{u}^m), \chi^m), \varepsilon(\boldsymbol{u}_t^m)) = (\boldsymbol{b}, \boldsymbol{u}_t^m).$$

Further, testing $(3.6)_2$ by μ^m yields

$$(3.12) \qquad (\chi_t^m, \mu^m) + \|\nabla\mu^m\|^2_{L_2(\Omega)} = 0.$$

Finally, testing $(3.6)_3$ by $-\ \chi_t^m(t)$ and integrating by parts, leads to

$$(3.13) \qquad -(\mu^m, \chi_t^m) + \frac{1}{2}\frac{d}{dt}\|\nabla\chi^m\|^2_{L_2(\Omega)} + (\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m), \chi_t^m) = 0.$$

Summing up (3.11), (3.12) and (3.13) we arrive at the following energy identity

$$(3.14) \qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\limits_\Omega |\boldsymbol{u}_t^m|^2 dx + \frac{d}{dt}\int\limits_\Omega \Big[W(\varepsilon(\boldsymbol{u}^m), \chi^m) + \psi(\chi^m) + \frac{1}{2}|\nabla\chi^m|\Big] dx \\ &+ \int\limits_\Omega |\nabla\mu^m|^2 dx = \int\limits_\Omega \boldsymbol{b}\cdot\boldsymbol{u}_t^m dx. \end{aligned}$$

Integration of (3.14) over $(0, t)$ gives

$$(3.15) \qquad \begin{aligned} &\frac{1}{2}\int\limits_\Omega |\boldsymbol{u}_t^m|^2 dx + \int\limits_\Omega f(\varepsilon(\boldsymbol{u}^m), \chi^m, \nabla\chi^m) dx + \int\limits_{\Omega^t} |\nabla\mu^m|^2 dx dt' \\ &= \frac{1}{2}\int\limits_\Omega |\boldsymbol{u}_1^m|^2 dx + \int\limits_\Omega f(\varepsilon(\boldsymbol{u}_0^m), \chi_0^m, \nabla\chi_0^m) dx + \int\limits_{\Omega^t} \boldsymbol{b}\cdot\boldsymbol{u}_{t'}^m dx dt', \end{aligned}$$

with $f(\varepsilon, \chi, \nabla\chi)$ defined by (2.4). Now, bearing in mind that $f(\varepsilon, \chi, \nabla\chi)$ satisfies the structure condition (2.9), we can estimate the left-hand side of (3.15) from below by

$$\frac{1}{2}\|\boldsymbol{u}_t^m\|^2_{L_2(\Omega)} + c_f(\|\varepsilon(\boldsymbol{u}^m)\|^2_{L_2(\Omega)} + \|\chi^m\|^4_{L_4(\Omega)} + \|\nabla\chi^m\|^2_{L_2(\Omega)}) + \|\nabla\mu^m\|^2_{L_2(\Omega^t)} - c'_f.$$

Further, in view of growth conditions (2.7) and the convergences (3.7), we have

$$\int\limits_\Omega f(\varepsilon(\boldsymbol{u}_0^m), \chi_0^m, \nabla\chi_0^m) dx \le c(\|\varepsilon(\boldsymbol{u}_0)\|^2_{L_2(\Omega)} + \|\chi_0\|^4_{L_4(\Omega)} + \|\nabla\chi_0\|^2_{L_2(\Omega)} + 1).$$

Thus the sum of the first two terms on the right-hand side of (3.15) is bounded from above by a constant depending on $\|\chi_0\|_{H^1(\Omega)}$, $\|\boldsymbol{u}_0\|_{H^1(\Omega)}$ and $\|\boldsymbol{u}_1\|_{L_2(\Omega)}$.

Finally, estimating the third term on the right-hand side of (3.15) by

$$\left|\int\limits_{\Omega^t} \boldsymbol{b}\cdot \boldsymbol{u}^m_{t'}\,dx dt'\right| \le \|\boldsymbol{u}^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))}\|\boldsymbol{b}\|_{L_1(0,t;L_2(\Omega))}$$

$$\le \frac{1}{4}\|\boldsymbol{u}^m_{t'}\|^2_{L_\infty(0,t;L_2(\Omega))} + \|\boldsymbol{b}\|^2_{L_1(0,t;L_2(\Omega))},$$

we arrive at the following uniform in m estimate

$$\begin{aligned}(3.16)\qquad &\frac{1}{4}\|\boldsymbol{u}^m_t\|^2_{L_2(\Omega)} + c_f(\|\varepsilon(\boldsymbol{u}^m)\|^2_{L_2(\Omega)} + \|\chi^m\|^4_{L_4(\Omega)} + \|\nabla\chi^m\|^2_{L_2(\Omega)})\\ &+\|\nabla\mu^m\|^2_{L_2(\Omega^t)} \le c \quad \text{for} \quad t\in(0,T],\end{aligned}$$

with constant c depending only on $\|\boldsymbol{u}_0\|_{H^1(\Omega)}$, $\|\boldsymbol{u}_1\|_{L_2(\Omega)}$, $\|\chi_0\|_{H^1(\Omega)}$, $\|\boldsymbol{b}\|_{L_1(0,t;L_2(\Omega))}$ and c'_f. This proves the assertion. □

3.3. Further estimates

Clearly, (3.9) implies that

$$(3.17)\qquad \|\chi^m\|_{L_\infty(0,T;H^1(\Omega))} \le c_1$$

with constant

$$c_1 = c(c_0,\Omega).$$

Hence, by the Sobolev imbedding,

$$(3.18)\qquad \|\chi^m\|_{L_\infty(0,T;L_6(\Omega))} \le c_1.$$

Further, since $\boldsymbol{u}^m = 0$ on S^T, it follows from (3.9) by Korn's inequality that

$$(3.19)\qquad \|\boldsymbol{u}^m\|_{L_\infty(0,T;H^1(\Omega))} \le c_1.$$

We note that setting $w_j = 1$ in $(3.6)_2$ (admissible by assumption) yields

$$(3.20)\qquad \frac{d}{dt}\int\limits_\Omega \chi^m dx = 0,$$

which shows that the mean value of χ^m is preserved

$$\int\limits_\Omega \chi^m(t)dx = \int\limits_\Omega \chi^m_0 dx \quad \text{for} \quad t\in[0,T].$$

This property will be used in later analysis. We remark also that thanks to (3.20) structure condition (2.9) on $f(\varepsilon,\chi,\nabla\chi)$ could be in fact repalced by a weaker one

$$f(\varepsilon,\chi,\nabla\chi) \ge c_f(|\varepsilon|^2 + |\nabla\chi|^2) - c'_f.$$

In such a case, by the Poincaré inequality, estimate $\|\nabla\chi\|_{L_\infty(0,T;L_2(\Omega))} \le c_0$ would still guarantee bounds (3.17) and (3.18) with a constant $c = c(c_0,\Omega,\int_\Omega \chi_0 dx)$.

On the basis of (3.9) and (3.18) we derive an additional estimate on μ^m.

Lemma 3.2. *Let assumptions of Lemma 3.1 be satisfied. Then, for $t \in (0,T]$,*

$$\|\mu^m\|_{L_2(0,t;H^1(\Omega))} \le c_2(t) \tag{3.21}$$

with constant

$$c_2(t) = c(c_1)t^{1/2}.$$

Proof. Setting $w_j = 1$ n $(3.6)_3$ (admissible by assumption) it follows that

$$\int_\Omega \mu^m dx = \int_\Omega [\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]dx.$$

Hence, using growth conditions (2.7) and estimates (3.9), (3.18) we obtain

$$\left| \int_\Omega \mu^m dx \right| \le c \int_\Omega (|\chi|^3 + |\varepsilon(\boldsymbol{u})| + 1)dx \le c(c_1) \quad \text{for a.a. } t \in (0,T]. \tag{3.22}$$

Consequently, by the Poincaré inequality, estimates (3.9) and (3.22) imply that

$$\begin{aligned} \|\mu^m\|_{L_2(\Omega^t)} &\le c\|\nabla\mu^m\|_{L_2(\Omega^t)} + c\Big[\int_0^t \Big(\int_\Omega \mu^m dx\Big)^2 dt'\Big]^{1/2} \\ &\le cc_0 + c(c_1)t^{1/2} \le c_2(t). \end{aligned}$$

This shows (3.21). □

By virtue of Lemma 3.2 we can deduce further estimates on χ^m.

Lemma 3.3. *Let assumptions of Lemma 3.1 hold. Then, for $t \in (0,T]$,*

$$\|\chi^m\|_{L_2(0,t;H^2_N(\Omega))} \le c_2(t). \tag{3.23}$$

Proof. In view of (3.2), identity $(3.6)_3$ implies that

$$(\mu^m, \Delta w_j) = (-\Delta\chi^m, \Delta w_j) + (\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m), \Delta w_j).$$

Testing the above equality by $\chi^m(t)$ and integrating with respect to t yields

$$\begin{aligned} \int_0^t \int_\Omega (\Delta\chi^m)^2 dx dt' &= -\int_0^t \int_\Omega \mu^m \Delta\chi^m dx dt' \\ &+ \int_0^t \int_\Omega [\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]\Delta\chi^m dx dt'. \end{aligned}$$

Now, using the Cauchy-Schwartz inequality, and then growth conditions (2.7) and estimates (3.9), (3.18), (3.21), we obtain

$$
\begin{aligned}
\|\Delta\chi^m\|_{L_2(\Omega^t)} &\le \|\mu^m\|_{L_2(\Omega^t)} + \|\psi'(\chi^m)\|_{L_2(\Omega^t)} + \|W_{,\chi}(\varepsilon(u^m),\chi^m)\|_{L_2(\Omega^t)} \\
&\le \|\mu^m\|_{L_2(\Omega^t)} + ct^{1/2}(\|\chi\|^3_{L_\infty(0,t;L_6(\Omega))} + \|\varepsilon(u^m)\|_{L_\infty(0,t;L_2(\Omega))} + 1) \le c_2(t).
\end{aligned}
\tag{3.24}
$$

On account of the ellipticity property of the Laplace operator (see e.g. [LU73], Chap. III. 8) we have

$$
\|\chi^m\|_{H^2(\Omega)} \le c\Big(\|\Delta\chi^m\|_{L_2(\Omega)} + \Big|\int_\Omega \chi^m dx\Big|\Big).
\tag{3.25}
$$

Hence, by (3.24), (3.20) and the convergences $(3.7)_3$, we conclude (3.23). □

Using standard duality arguments we shall estimate also time derivatives u^m_{tt} and χ^m_t.

Lemma 3.4. *Let assumptions of Lemma 3.1 hold, and $b \in L_2(\Omega^T)$. Then, for $t \in (0,T]$,*

$$
\begin{aligned}
&\|u^m_{t't'}\|_{L_2(0,t;(H^1_0(\Omega))')} \le c_3(t), \\
&\|\chi^m_{t'}\|_{L_2(0,t;(H^1(\Omega))')} \le c_0,
\end{aligned}
\tag{3.26}
$$

where $L_2(0,t;(H^1(\Omega))')$

$$
c_3(t) = c(c_0, \|b\|_{L_2(\Omega^t)})t^{1/2}.
$$

Proof. For $\eta \in L_2(0,T;H^1_0(\Omega))$, we test $(3.6)_1$ by $\eta^m = P^m\eta$, where P^m denotes the projection defined by

$$
P^m\eta = \sum_{i=1}^m (\eta, v_i)v_i.
\tag{3.27}
$$

Then, using the Cauchy-Schwartz inequality, and recalling estimate (3.9), we obtain

$$
\begin{aligned}
&\Big|\int_0^t (u^m_{t't'},\eta)dt'\Big| = \Big|\int_0^t (u^m_{t't'},P^m\eta)dt'\Big| \\
&= \Big|\int_0^t [-(A\varepsilon(u^m),\varepsilon(P^m\eta)) + (z'(\chi^m)B\nabla\chi^m + b, P^m\eta)]dt'\Big| \\
&\le c[\|\varepsilon(u^m)\|_{L_2(\Omega^t)}\|\nabla P^m\eta\|_{L_2(\Omega^t)} + (\|\nabla\chi^m\|_{L_2(\Omega^t)} + \|b\|_{L_2(\Omega^t)})\|P^m\eta\|_{L_2(\Omega^t)}] \\
&\le c(c_0t^{1/2} + \|b\|_{L_2(\Omega^t)})\|P^m\eta\|_{L_2(0,t;H^1(\Omega))} \\
&\le c_3(t)\|\eta\|_{L_2(0,t;H^1(\Omega))} \quad \text{for all} \quad \eta \in L_2(0,t;H^1_0(\Omega)).
\end{aligned}
$$

This shows $(3.26)_1$. Similarly, for $\xi \in L_2(0,T;H^1(\Omega))$, we test $(3.6)_2$ by

$$\xi^m = P^m\xi = \sum_{i=1}^{m}(\xi, w_i)w_i, \tag{3.28}$$

to obtain

$$\begin{aligned}\Big|\int_0^t (\chi^m_{t'},\xi)dt'\Big| &= \Big|\int_0^t (\chi^m_{t'},P^m\xi)dt'\Big| = \Big|\int_0^t (\nabla\mu^m,\nabla P^m\xi)dt'\Big| \\ &\le \|\nabla\mu^m\|_{L_2(\Omega^t)}\|\nabla P^m\xi\|_{L_2(\Omega^t)} \le c_0\|\xi\|_{L_2(0,t;H^1(\Omega))},\end{aligned}$$

where in the last inequality we used (3.9). This implies $(3.26)_2$ and completes the proof.
□

4. Time regularity estimates

In this section we derive uniform in m time-regularity estimates for solutions of approximate system (3.6). These estimates result from time-differentiated version of (3.6) and lead to the existence result of Theorem 2.2. Throughout this section we assume that the domain Ω has the boundary S of class C^4.

Let us differentiate system $(3.6)_1$–$(3.6)_3$ with respect to t and rewrite it in the following form:

$$\begin{aligned}&(\boldsymbol{u}^m_{ttt},\boldsymbol{v}_j) + (A(\varepsilon(\boldsymbol{u}^m_t)),\varepsilon(\boldsymbol{v}_j)) = ([z'(\chi^m)B\nabla\chi^m]_{,t} + \boldsymbol{b}_t,\boldsymbol{v}_j),\\ &(\chi^m_{tt},w_j) - (\mu^m_t,\Delta w_j) = 0,\\ &(\mu^m_t,w_j) = -(\Delta\chi^m_t,w_j) + ([\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m),\chi^m)]_{,t},w_j),\\ &\qquad j = 1,\dots,m,\end{aligned} \tag{4.1}$$

where the explicit expressions for $[z'(\chi)B\nabla\chi]_{,t}$ and $[\psi'(\chi) + W_{,\chi}(\varepsilon(\boldsymbol{u}),\chi)]_{,t}$ are given in (2.21). The above system is considered with the initial conditions

$$\begin{aligned}&\boldsymbol{u}^m(0) = \boldsymbol{u}^m_0, \quad \boldsymbol{u}^m_t(0) = \boldsymbol{u}^m_1, \quad \chi^m(0) = \chi_0,\\ &\boldsymbol{u}^m_{tt}(0) = \boldsymbol{u}^m_2, \quad \chi^m_t(0) = \chi^m_1, \quad \mu^m(0) = \mu^m_0,\end{aligned} \tag{4.2}$$

where $\boldsymbol{u}^m_2 \in \boldsymbol{V}_{0m}$ and $\chi^m_1, \mu^m_0 \in V_m$ are the projections respectively of the data $\boldsymbol{u}_2, \chi_1$ and μ_0, defined in (2.16).
We assume that the following convergences in the strong sense are satisfied:

$$\begin{aligned}&\boldsymbol{u}^m_0 \to \boldsymbol{u}_0 \ \text{in}\ \boldsymbol{H}^3(\Omega)\cap\boldsymbol{H}^1_0(\Omega), && \boldsymbol{u}^m_1 \to \boldsymbol{u}_1 \ \text{in}\ \boldsymbol{H}^1_0(\Omega),\\ &\chi^m_0 \to \chi_0 \ \text{in}\ H^4(\Omega)\cap H^2_N(\Omega), && \boldsymbol{u}^m_2 \to \boldsymbol{u}_2 \ \text{in}\ L_2(\Omega),\\ &\chi^m_1 \to \chi_1 \ \text{in}\ L_2(\Omega), && \mu^m_0 \to \mu_0 \ \text{in}\ H^2_N(\Omega).\end{aligned} \tag{4.3}$$

4.1. The basic estimate

Lemma 4.1. *Let (A1)–(A5) hold, the boundary of the domain Ω be of class C^4, the function $z:\mathbb{R}\to[0,1]$ be of class C^2 with property (2.17) and the data satisfy (2.18). Then a solution $(\boldsymbol{u}^m,\chi^m,\mu^m)$ of approximate problem (3.6) satisfies system (4.1) with initial conditions (4.2). Moreover, $(\boldsymbol{u}^m,\chi^m,\mu^m)$ satisfy the estimates in Lemmas 3.1–3.4, and*

$$
\begin{aligned}
&\|\boldsymbol{u}^m_{t't'}\|_{L_\infty(0,t;L_2(\Omega))}+\|\boldsymbol{u}^m_{t'}\|_{L_\infty(0,t;H^1(\Omega))}\le c_5(t),\\
&\|\chi^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))}+\|\chi^m_{t'}\|_{L_2(0,t;H^2_N(\Omega))}\le c_4(t)
\end{aligned}
\tag{4.4}
$$

for $t\in(0,T]$, with constants $c_4(t),c_5(t)$ (independent of m) given by

$$
\begin{aligned}
&c_4(t)=c(t^{1/2}E_1(t)+\|\chi_1\|_{L_2(\Omega)})[\exp a(t)]^{1/2},\\
&c_5(t)=t^{1/2}c_4(t),
\end{aligned}
$$

where

$$
\begin{aligned}
&E_1(t)=t^{1/2}\|\boldsymbol{b}_{t'}\|_{L_2(\Omega^t)}+\|\boldsymbol{u}_2\|_{L_2(\Omega)}+\|\varepsilon(\boldsymbol{u}_1)\|_{L_2(\Omega)},\\
&a(t)=c(c_0)t^8\exp(ct).
\end{aligned}
$$

Proof. In the first step we estimate $\boldsymbol{u}^m_t$ in terms of the $L_2(0,t;H^2(\Omega))$ – norm of χ^m_t. Testing $(4.1)_1$ by $\boldsymbol{u}^m_{tt}(t)$, we obtain

$$
\frac{1}{2}\frac{d}{dt}\int\limits_\Omega(|\boldsymbol{u}^m_{tt}|^2+\varepsilon(\boldsymbol{u}^m_t)\cdot A\varepsilon(\boldsymbol{u}^m_t))dx=\int\limits_\Omega([z'(\chi^m)B\nabla\chi^m]_{,t}+\boldsymbol{b}_t)\cdot\boldsymbol{u}^m_{tt}dx.
$$

Hence, by the Cauchy-Schwartz inequality,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg[\bigg(\int\limits_\Omega(|\boldsymbol{u}^m_{tt}|^2+\varepsilon(\boldsymbol{u}^m_t)\cdot A\varepsilon(\boldsymbol{u}^m_t))dx\bigg)^{1/2}\Bigg]^2\\
&\le c\bigg(\int\limits_\Omega((\chi^m_t)^2|\nabla\chi^m|^2+|\nabla\chi^m_t|^2+|\boldsymbol{b}_t|^2)dx\bigg)^{1/2}\bigg(\int\limits_\Omega(|\boldsymbol{u}^m_{tt}|^2+\varepsilon(\boldsymbol{u}^m_t)\cdot A\varepsilon(\boldsymbol{u}^m_t))dx\bigg)^{1/2},
\end{aligned}
$$

so

$$
\frac{d}{dt}\bigg(\int\limits_\Omega(|\boldsymbol{u}^m_{tt}|^2+\varepsilon(\boldsymbol{u}^m_t)\cdot A\varepsilon(\boldsymbol{u}^m_t))dx\bigg)^{1/2}\le c\bigg(\int\limits_\Omega((\chi^m_t)^2|\nabla\chi^m|^2+|\nabla\chi^m_t|^2+|\boldsymbol{b}_t|^2)dx\bigg)^{1/2}.
\tag{4.5}
$$

Integrating (4.5) with respect to t and using the coercivity of A (see (2.2)), it follows that

(4.6)

$$\int_\Omega (|u_{tt}^m|^2 + |\varepsilon(u_t^m)|^2)dx \le c\Big[\int_0^t \Big(\int_\Omega ((\chi_{t'}^m)^2|\nabla\chi^m|^2 + |\nabla\chi_{t'}^m|^2 + |b_{t'}|^2)dx\Big)^{1/2} dt'\Big]^2$$

$$+ \int_\Omega (|u_{tt}^m(0)|^2 + |\varepsilon(u_t^m(0))|^2)dx$$

$$\le ct \int_{\Omega^t} ((\chi_{t'}^m)^2|\nabla\chi^m|^2 + |\nabla\chi_{t'}^m|^2)dxdt' + c(E_1^m(t))^2,$$

where

$$E_1^m(t) := t^{1/2}\|b_t\|_{L_2(\Omega^t)} + \|u_2^m\|_{L_2(\Omega)} + \|\varepsilon(u_1^m)\|_{L_2(\Omega)}.$$

Clearly, due to convergences (4.3), $E_1^m(t) \le cE_1(t)$.
We shall estimate the first two terms on the right-hand side of (4.6). On account of Lemma 3.1, we obtain

$$\text{(4.7)}\qquad \begin{aligned}\int_{\Omega^t} (\chi_{t'}^m)^2|\nabla\chi^m|^2 dxdt' &\le \sup_{t'}\int_\Omega |\nabla\chi^m|^2 dx \int_0^t \|\chi_{t'}^m\|_{L_\infty(\Omega)}^2 dt' \\ &\le c_0\int_0^t \|\chi_{t'}^m\|_{L_\infty(\Omega)}^2 dt' = I_1.\end{aligned}$$

Now, applying the interpolation inequality (see e.q. [BIN96], Chap. III, Sec. 10)

$$\|\chi_t^m\|_{L_\infty(\Omega)} \le \varepsilon^{1-\varkappa_1}\|\nabla^2\chi_t^m\|_{L_2(\Omega)} + c\varepsilon^{-\varkappa_1}\|\chi_t^m\|_{L_2(\Omega)}$$

with $\varkappa_1 = 3/4$, $\varepsilon > 0$, and setting $\delta_1 = \varepsilon^{1/4}$, yields

$$\text{(4.8)}\qquad I_1 \le \delta_1\int_0^t \|\nabla^2\chi_{t'}^m\|_{L_2(\Omega)}^2 dt' + c(c_1)\delta_1^{-3}\int_0^t \|\chi_{t'}^m\|_{L_2(\Omega)}^2 dt'.$$

Similarly,

$$\text{(4.9)}\qquad \int_{\Omega^t} |\nabla\chi_{t'}^m|^2 dxdt' \le \delta_2\int_0^t \|\nabla^2\chi_{t'}^m\|_{L_2(\Omega)}^2 dt' + c\delta_2^{-1}\int_0^t \|\chi_{t'}^m\|_{L_2(\Omega)}^2 dt',$$

where the interpolation inequality

$$\|\nabla\chi_t^m\|_{L_2(\Omega)} \le \varepsilon^{1-\varkappa_2}\|\nabla^2\chi_t^m\|_{L_2(\Omega)} + c\varepsilon^{-\varkappa_2}\|\chi_t^m\|_{L_2(\Omega)}$$

with $\varkappa_2 = 1/2$, $\varepsilon > 0$ and $\delta_2 = \varepsilon^{1/2}$, was applied.

Using (4.7)–(4.9) in (4.6) yields

$$\int_\Omega (|u_{tt}^m|^2 + |\varepsilon(u_t^m)|^2)dx \le ct(\delta_1+\delta_2)\int_0^t \|\nabla^2\chi_{t'}^m\|_{L_2(\Omega)}^2 dt'$$

$$+ c(c_0)t(\delta_1^{-3}+\delta_2^{-1})\int_0^t \|\chi_{t'}^m\|_{L_2(\Omega)}^2 dt' + c(E_1(t))^2.$$

Hence, assuming $\delta_1=\delta_2$ and choosing $\delta = ct\delta_1$ we arrive at the estimate

$$\begin{aligned} &\|u_{tt}^m\|_{L_2(\Omega)}^2 + \|\varepsilon(u_t^m)\|_{L_2(\Omega)}^2 \\ &\le \delta\int_0^t \|\nabla^2\chi_{t'}^m\|_{L_2(\Omega)}^2 dt' + c(1/\delta, c_0, t)\int_0^t \|\chi_{t'}^m\|_{L_2(\Omega)}^2 dt' \\ &\quad + c(E_1(t))^2 \quad \text{for} \quad t\in(0,T], \end{aligned} \tag{4.10}$$

where

$$c(1/\delta, c_0, t) = c(c_0)\delta^{-3}t^4, \quad \delta>0 \ \ \text{(arbitrary)}.$$

In the second step we consider system $(4.1)_2$, $(4.1)_3$ which on account of (3.2) can be rewritten in the form of the following equation:

$$(\chi_{tt}^m, w_j) = -(\Delta\chi_t^m, \Delta w_j) + ([\psi'(\chi^m) + W_{,\chi}(\varepsilon(u^m),\chi^m)]_{,t}, \Delta w_j). \tag{4.11}$$

Testing (4.11) by $\chi_t^m(t)$ gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\chi_t^m)^2 dx + \int_\Omega (\Delta\chi_t^m)^2 dx = \int_\Omega [\psi'(\chi^m) + W_{,\chi}(\varepsilon(u^m),\chi^m)]_{,t}\Delta\chi_t^m dx.$$

Hence, by the Young inequality, it follows that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (\chi_t^m)^2 dx + \int_\Omega (\Delta\chi_t^m)^2 dx \le \int_\Omega [\psi'(\chi^m) + W_{,\chi}(\varepsilon(u^m),\chi^m)]_{,t}^2 dx \\ &\le c\int_\Omega [(\chi^m)^4(\chi_t^m)^2 + (\chi_t^m)^2|\varepsilon(u^m)|^2 + (\chi_t^m)^2 + |\varepsilon(u_t^m)|^2]dx, \end{aligned} \tag{4.12}$$

where in the last inequality we used identity $(2.21)_2$ and the assumptions on ψ and z.

Let us examine the first two terms on the right-hand side of (4.12). Using the Hölder inequality and then recalling estimate (3.18), we obtain

$$\begin{aligned} &\int_\Omega (\chi^m)^4(\chi_t^m)^2 dx \le \sup_t \|\chi^m\|_{L_6(\Omega)}^4 \|\chi_t^m\|_{L_6(\Omega)}^2 \\ &\le c_1\|\chi_t^m\|_{L_6(\Omega)}^2 \le \delta_3\|\nabla^2\chi_t^m\|_{L_2(\Omega)}^2 + c(1/\delta_3)\|\chi_t^m\|_{L_2(\Omega)}^2, \quad \delta_3>0, \end{aligned} \tag{4.13}$$

where in the last line an interpolation inequality was used.

Similarly, an application of the Hölder inequality, estimate (3.9) and an interpolation inequality to the second term on the right-hand side of (4.12) yields

$$\begin{aligned}\int_\Omega (\chi_t^m)^2|\varepsilon(u^m)|^2dx &\le \sup_t \|\varepsilon(u^m)\|^2_{L_2(\Omega)}\|\chi_t^m\|^2_{L_\infty(\Omega)}\\ &\le c_0\|\chi_t^m\|^2_{L_\infty(\Omega)}\\ &\le \delta_4\|\nabla^2\chi_t^m\|^2_{L_2(\Omega)} + c(1/\delta_4)\|\chi_t^m\|^2_{L_2(\Omega)}, \quad \delta_4>0.\end{aligned} \tag{4.14}$$

Now, using the inequality

$$\|\chi_t^m\|_{H^2_N(\Omega)} \le c\|\Delta\chi_t^m\|_{L_2(\Omega)}, \tag{4.15}$$

which holds true because $\int_\Omega \chi_t^m dx = 0$ (see (3.20)), applying (4.13) and (4.14) in (4.12), and choosing δ_3, δ_4 sufficiently small, we conclude that

$$\frac{d}{dt}\|\chi_t^m\|^2_{L_2(\Omega)} + \|\chi_t^m\|^2_{H^2_N(\Omega)} \le c(\|\chi_t^m\|^2_{L_2(\Omega)} + \|\varepsilon(u_t^m\|^2_{L_2(\Omega)}). \tag{4.16}$$

At this point we apply estimate (4.10) to the second term on the right-hand side of (4.16). This leads to

$$\begin{aligned}&\frac{d}{dt}\|\chi_t^m\|^2_{L_2(\Omega)} + \|\chi_t^m\|^2_{H^2_N(\Omega)}\\ &\le c\|\chi_t^m\|^2_{L_2(\Omega)} + \delta\int_0^t \|\nabla^2\chi_{t'}^m\|^2_{L_2(\Omega)}dt' + c(c_0)\delta^{-3}t^4\int_0^t\|\chi_{t'}^m\|^2_{L_2(\Omega)}dt'\\ &\quad + c(E_1(t))^2 \quad \text{for} \quad t\in(0,T].\end{aligned} \tag{4.17}$$

Multiplying (4.17) by e^{-ct}, integrating with respect to t, and using that on account of (4.3), $\|\chi_1^m\|_{L_2(\Omega)} \le c\|\chi_1\|_{L_2(\Omega)}$, we arrive at

$$\begin{aligned}&e^{-ct}\|\chi_t^m(t)\|^2_{L_2(\Omega)} + \int_0^t e^{-ct'}\|\chi_{t'}^m(t')\|^2_{H^2_N(\Omega)}dt' \le \delta\int_0^t e^{-ct'}\int_0^{t'}\|\nabla^2\chi_{t''}^m(t'')\|^2_{L_2(\Omega)}dt''dt'\\ &\quad + c(c_0)\delta^{-3}\int_0^t (t')^4e^{-ct'}\int_0^{t'}\|\chi_{t''}^m(t'')\|^2_{L_2(\Omega)}dt''dt' + c\int_0^t e^{-ct'}(E_1(t'))^2dt' + \|\chi_1\|^2_{L_2(\Omega)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\|\chi_t^m(t)\|^2_{L_2(\Omega)} + \int_0^t\|\chi_{t'}^m(t')\|^2_{H^2_N(\Omega)}dt' \le \delta t e^{ct}\int_0^t \|\chi_{t'}^m(t')\|^2_{H^2_N(\Omega)}dt'\\ &\quad + c(c_0)\delta^{-3}t^5e^{ct}\int_0^t\|\chi_{t'}^m(t')\|^2_{L_2(\Omega)}dt' + cte^{ct}(E_1(t))^2 + e^{ct}\|\chi_1\|^2_{L_2(\Omega)}.\end{aligned} \tag{4.18}$$

Choosing $\delta = t^{-1}e^{-ct}/2$, we obtain

$$\|\chi_t^m(t)\|^2_{L_2(\Omega)} + \int_0^t \|\chi_{t'}^m(t')\|^2_{H^2_N(\Omega)}dt'$$
$$\leq c(c_0)t^8 e^{ct}\int_0^t \|\chi_{t'}^m(t')\|^2_{L_2(\Omega)}dt' + cte^{ct}(E_1(t))^2 + e^{ct}\|\chi_1\|^2_{L_2(\Omega)}.$$

Now, the application of the Gronwall lemma yields

$$\begin{aligned}&\|\chi_t^m(t)\|^2_{L_2(\Omega)} + \int_0^t \|\chi_{t'}^m(t')\|^2_{H^2_N(\Omega)}dt' \\ &\leq c(t(E_1(t))^2 + \|\chi_1\|_{L_2(\Omega)})\exp(ct + c(c_0)t^8\exp(ct)) \\ &\leq c_4^2(t) \quad \text{for a.a. } t \in (0,T].\end{aligned} \tag{4.19}$$

This proves estimate $(4.4)_2$. Applying (4.19) in (4.10) and setting there $\delta = t$ we conclude, by virtue of Korn's inequality, estimate $(4.4)_1$. Thereby the proof is completed. □

4.2. Further estimates

Firstly, we note that in view of the inequality

$$|\chi_{xx}(t) - \chi_{xx}(t')| \leq |t-t'|^{1/2}\bigg(\int_{t'}^t \chi^2_{t''xx}dt''\bigg)^{1/2},$$

estimate $(4.4)_2$ implies that $\chi^m \in C^{1/2}([0,T]; H^2_N(\Omega))$, and

$$\|\chi^m\|_{C^{1/2}([0,t];H^2_N(\Omega))} \leq c_4(t). \tag{4.20}$$

Next, we prove the following

Lemma 4.2. *Let assumptions of Lemma 4.1 hold. Then, for $t \in (0,T]$,*

$$\|\boldsymbol{u}^m\|_{L_\infty(0,t;H^2(\Omega))} \leq c_5(t). \tag{4.21}$$

Proof. Using (3.1) we rewrite $(3.6)_1$ in the form

$$(\boldsymbol{u}^m_{tt}, Q\boldsymbol{v}_j) - (Q\boldsymbol{u}^m, Q\boldsymbol{v}_j) = (z'(\chi^m)B\nabla\chi^m + \boldsymbol{b}, Q\boldsymbol{v}_j).$$

Testing this equality by $\boldsymbol{u}^m(t)$ gives

$$\|\boldsymbol{Q}\boldsymbol{u}^m\|^2_{L_2(\Omega)} = (\boldsymbol{u}^m_{tt} - z'(\chi^m)B\nabla\chi^m + \boldsymbol{b}, \boldsymbol{Q}\boldsymbol{u}^m) \quad \text{for a.a. } t \in (0,T].$$

Hence, using the Cauchy-Schwartz inequality and then recalling estimates (3.9) and $(4.4)_1$, we obtain

$$\begin{aligned}\|\boldsymbol{Q}\boldsymbol{u}^m\|_{L_\infty(0,t;L_2(\Omega))} &\leq c(\|\boldsymbol{u}^m_{t't'}\|_{L_\infty(0,t;L_2(\Omega))} + \|\nabla\chi^m\|_{L_\infty(0,t;L_2(\Omega))} \\ &+ \|\boldsymbol{b}\|_{L_\infty(0,t;L_2(\Omega))}) \leq c(c_5(t) + c_0 + 1) \leq c_5(t).\end{aligned}$$

This, by the ellipticity property of $\boldsymbol{Q}$ (see (2.3)), implies (4.21). □

The next result provides an additional regularity estimate for μ^m.

Lemma 4.3. *Under assumptions of Lemma 4.1,*

$$\begin{aligned}&\|\mu^m\|_{L_\infty(0,t;H^2_N(\Omega))} \leq c_4(t), \\ &\|\mu^m_{t'}\|_{L_2(\Omega^t)} \leq t^{1/2}c_5(t), \quad t \in (0,T].\end{aligned} \tag{4.22}$$

Proof. Using (3.2) we rewrite identity $(3.6)_2$ in the form

$$(\chi^m_t, \Delta w_j) = (\Delta\mu^m, \Delta w_j).$$

Testing this equality by $\mu^m(t)$ and using the Cauchy-Schwartz inequality yields

$$\|\Delta\mu^m\|_{L_2(\Omega)} \leq \|\chi^m_t\|_{L_2(\Omega)} \quad \text{for a.a. } t \in (0,T].$$

Consequently, due to estimate $(4.4)_2$,

$$\|\Delta\mu^m\|_{L_\infty(0,t;L_2(\Omega))} \leq \|\chi^m_{t'}\|_{L_\infty(0,t;L_2(\Omega))} \leq c_4(t). \tag{4.23}$$

Thus, recalling bound (3.22) on the mean value of μ^m, estimate $(4.22)_1$ follows from (4.23) on account of the ellipticity property of the Laplace operator.

To show $(4.22)_2$ we test identity $(4.1)_3$ by $\mu^m_t(t)$. Then, with the help of the Cauchy-Schwartz inequality, it follows that

$$\|\mu^m_t\|_{L_2(\Omega)} \leq \|\Delta\chi^m_t\|_{L_2(\Omega)} + \|[\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]_{,t}\|_{L_2(\Omega)}. \tag{4.24}$$

The second term on the right-hand side of (4.24) can be estimated with the help of bounds (4.12)–(4.14) (with $\delta_3 = \delta_4 = 1$) in the proof of Lemma 4.1. Consequently, we obtain

$$\begin{aligned}\|\mu^m_t\|_{L_2(\Omega^t)} &\leq c(\|\Delta\chi^m_t\|_{L_2(\Omega^t)} + \|\nabla^2\chi^m_t\|_{L_2(\Omega^t)} + \|\chi^m_t\|_{L_2(\Omega^t)} \\ &+ \|\varepsilon(\boldsymbol{u}^m_t)\|_{L_2(\Omega^t)}) \leq c(c_4(t) + t^{1/2}c_5(t)) \leq t^{1/2}c_5(t),\end{aligned}$$

where in the last line Lemma 4.1 was used. This completes the proof. □

Finally, we estimate time derivatives u_{ttt}^m and χ_{tt}^m.

Lemma 4.4. *Under assumptions of Lemma 4.1,*

$$
(4.25)\qquad
\begin{aligned}
&\|u_{t't't'}^m\|_{L_2(0,t;(H_0^1(\Omega))')} \le t^{1/2}c_5(t),\\
&\|\chi_{t't'}^m\|_{L_2(0,t;(H_N^2(\Omega))')} \le t^{1/2}c_5(t), \quad t \in (0,T].
\end{aligned}
$$

Proof. We proceed similarly as in Lemma 3.4. For $\eta \in L_2(0,T; H_0^1(\Omega))$ we test $(4.1)_1$ by $\eta^m = P^m\eta$, where the projectin P^m is defined by (3.27). Then

$$
\begin{aligned}
\Big|\int_0^t (u_{t't't'}^m, \eta)dt'\Big| &= \Big|\int_0^t \{-(A\varepsilon(u_{t'}^m), \varepsilon(P^m\eta)) + ([z'(\chi^m)B\nabla\chi^m + b]_{,t'}, P^m\eta)\}dt'\Big|\\
&\le c[\|\varepsilon(u_{t'}^m)\|_{L_2(\Omega^t)}\|\nabla P^m\eta\|_{L_2(\Omega^t)}\\
&\quad + (\|\chi_{t'}^m\nabla\chi^m\|_{L_2(\Omega^t)} + \|\nabla\chi_{t'}^m\|_{L_2(\Omega^t)} + \|b_{t'}\|_{L_2(\Omega^t)})\|P^m\eta\|_{L_2(\Omega^t)}].
\end{aligned}
$$

Hence, recalling Lemmas 3.1, 4.1, and the estimate

$$
\|\chi_{t'}^m\nabla\chi^m\|_{L_2(\Omega^t)} \le \|\chi_{t'}^m\|_{L_2(0,t;L_\infty(\Omega))}\|\nabla\chi^m\|_{L_\infty(0,t;L_2(\Omega))} \le c_0c_4(t),
$$

it follows that

$$
\begin{aligned}
\Big|\int_0^t (u_{t't't'}^m, \eta)dt'\Big| &\le c(t^{1/2}c_5(t) + c_4(t) + \|b_{t'}\|_{L_2(\Omega^t)})\|P^m\eta\|_{L_2(0,t;H^1(\Omega))}\\
&\le t^{1/2}c_5(t)\|\eta\|_{L_2(0,t;H^1(\Omega))} \quad \text{for all} \quad \eta \in L_2(0,t; H_0^1(\Omega)).
\end{aligned}
$$

This shows $(4.25)_1$. Similarly, for any $\xi \in L_2(0,T; H_N^2(\Omega))$, testing $(4.1)_2$ by $\xi^m = P^m\xi$, where the projection P^m is defined by (3.28), we obtain

$$
\begin{aligned}
\Big|\int_0^t (\chi_{t't'}^m, \xi)dt'\Big| &= \Big|\int_0^t (\mu_{t'}^m, \Delta P^m\xi)dt'\Big| \le \|\mu_{t'}^m\|_{L_2(\Omega^t)}\|\Delta P^m\xi\|_{L_2(\Omega^t)}\\
&\le t^{1/2}c_5(t)\|\xi\|_{L_2(0,t;H_N^2(\Omega))},
\end{aligned}
$$

where in the last inequality Lemma 4.3 was applied. This shows $(4.25)_2$. □

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5. Proof of Theorem 2.1

From Lemmas 3.2–3.4 it follows that there exists a triple $(\boldsymbol{u}, \chi, \mu)$ with

$$
(5.1)\qquad
\begin{aligned}
&\boldsymbol{u} \in L_\infty(0,T;\boldsymbol{H}^1_0(\Omega)), \quad \boldsymbol{u}_t \in L_\infty(0,T;\boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_{tt} \in L_2(0,T;(\boldsymbol{H}^1_0(\Omega))'),\\
&\chi \in L_\infty(0,T;H^1(\Omega)) \cap L_2(0,T;H^2_N(\Omega)), \quad \chi_t \in L_2(0,T;(H^1(\Omega))'),\\
&\mu \in L_2(0,T;H^1(\Omega)),
\end{aligned}
$$

and a subsequence of solutions $(\boldsymbol{u}^m, \chi^m, \mu^m)$ to problem (3.6) (which we still denote by the same indices) such that as $m \to \infty$:

$$
(5.2)\qquad
\begin{aligned}
\boldsymbol{u}^m &\to \boldsymbol{u} && \text{weakly} -^* \text{ in } L_\infty(0,T;\boldsymbol{H}^1_0(\Omega)),\\
\boldsymbol{u}^m_t &\to \boldsymbol{u}_t && \text{weakly} -^* \text{ in } L_\infty(0,T;\boldsymbol{L}_2(\Omega)),\\
\boldsymbol{u}^m_{tt} &\to \boldsymbol{u}_{tt} && \text{weakly in } L_2(0,T;(\boldsymbol{H}^1_0(\Omega))'),\\
\chi^m &\to \chi && \text{weakly} -^* \text{ in } L_\infty(0,T;H^1(\Omega)) \text{ and}\\
& && \text{weakly in } L_2(0,T;H^2_N(\Omega)),\\
\chi^m_t &\to \chi_t && \text{weakly in } L_2(0,T;(H^1(\Omega))'),\\
\mu^m &\to \mu && \text{weakly in } L_2(0,T;H^1(\Omega)).
\end{aligned}
$$

Using the compactness results (see e.g. Lions [Lions69], Simon [Sim87], Sec. 8) it follows that for a subsequence (still denoted by the same indices)

$$
(5.3)\qquad
\begin{aligned}
\boldsymbol{u}^m &\to \boldsymbol{u} && \text{strongly in } L_2(0,T;\boldsymbol{L}_q(\Omega)) \cap C([0,T];\boldsymbol{L}_q(\Omega)), \quad q<6,\\
& && \text{and a.e. in } \Omega^T,\\
\boldsymbol{u}^m_t &\to \boldsymbol{u}_t && \text{strongly in } C([0,T];(\boldsymbol{H}^1_0(\Omega))'),\\
\chi^m &\to \chi && \text{strongly in } L_2(0,T;H^1(\Omega)) \cap C([0,T];L_2(\Omega))\\
& && \text{and a.e. in } \Omega^T.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\boldsymbol{u}^m(0) = \boldsymbol{u}^m_0 &\to \boldsymbol{u}(0) && \text{strongly in } \boldsymbol{L}_q(\Omega), \quad q<6,\\
\boldsymbol{u}^m_t(0) = \boldsymbol{u}^m_1 &\to \boldsymbol{u}_t(0) && \text{strongly in } (\boldsymbol{H}^1_0(\Omega))',\\
\chi^m(0) = \chi^m_0 &\to \chi(0) && \text{strongly in } L_2(\Omega),
\end{aligned}
$$

what together with convergences (3.7) implies that

$$
(5.4)\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \chi(0) = \chi_0.
$$

The relations (5.1) and (5.4) imply assertion (2.13) of the theorem.

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Now, let us introduce the following weak formulation of (3.6):

$$\int_0^T \langle \boldsymbol{u}^m_{tt}, \boldsymbol{\eta}\rangle_{(\boldsymbol{H}^1_0(\Omega))',\boldsymbol{H}^1_0(\Omega)}dt + \int_0^T (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))dt$$

$$= \int_0^T (z'(\chi^m)\boldsymbol{B}\nabla\chi^m + \boldsymbol{b}, \boldsymbol{\eta})dt \ \forall \boldsymbol{\eta} \in L_2(0,T;\boldsymbol{V}_{0m}),$$

$$\int_0^T \langle \chi^m_t, \xi\rangle_{(H^1(\Omega))',H^1(\Omega)}dt + \int_0^T (\nabla\mu^m, \nabla\xi)dt = 0 \tag{5.5}$$

$$\forall \xi \in L_2(0,T;V_m),$$

$$\int_0^T (\mu^m, \zeta)dt = -\int_0^T (\Delta\chi^m, \zeta)dt + \int_0^T (\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m), \zeta)dt$$

$$\forall \zeta \in L_2(0,T;V_m).$$

To pass to the limit $m \to \infty$ in identities (5.5) we follow the standard procedure (see e.g. Lions-Magenes [LionsMag72]). Namely, we fix $m = m_0 \in \mathbb{N}$ in the spaces of test functions $\boldsymbol{\eta}, \xi, \zeta$ and take subsequences (5.2) with $m \geq m_0$. Clearly, by virtue of the weak convergences (5.2), the linear terms in (5.5) converge to the corresponding limits. Thus, it remains to examine the convergence of the nonlinear terms $z'(\chi^m)\boldsymbol{B}\nabla\chi^m$ and $\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)$.

Recalling the growth conditions (2.7), and using the energy bounds (3.9), (3.18), it follows that

$$\begin{aligned}
&\|z'(\chi^m)\boldsymbol{B}\nabla\chi^m\|_{L_\infty(0,T;L_2(\Omega))} \leq c\|\nabla\chi^m\|_{L_\infty(0,T;L_2(\Omega))} \leq cc_0,\\
&\|\psi'(\chi^m)\|_{L_\infty(0,T;L_2(\Omega))} \leq c(\|\chi^m\|^3_{L_\infty(0,T;L_6(\Omega)} + 1) \leq c(c_1),\\
&\|W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)\|_{L_\infty(0,T;L_2(\Omega))} \leq c(\|\boldsymbol{\varepsilon}(\boldsymbol{u}^m)\|_{L_\infty(0,T;L_2(\Omega))} + 1) \leq c(c_0).
\end{aligned} \tag{5.6}$$

Thanks to these uniform in m estimates and the pointwise convergences (5.3) we can apply the standard nonlinear convergence lemma (see Lions [Lions69], Chapter 1, Lemma 1.3) to conclude that

(5.7)

$$\begin{aligned}
&z'(\chi^m)\boldsymbol{B}\nabla\chi^m \to z(\chi)\boldsymbol{B}\nabla\chi && \text{weakly} -^* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
&\psi'(\chi^m) = (\chi^m)^3 - \chi^m \to \chi^3 - \chi = \psi(\chi) && \text{weakly} -^* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
&W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m) = z'(\chi^m)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}^m) + Dz(\chi^m) + E) \to &&\\
&\quad z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E) = W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) && \text{weakly} -^* \text{ in } L_\infty(0,T;L_2(\Omega)).
\end{aligned}$$

Consequently, passing to the limit in (5.5) for a subsequence $m_0 \leq m \to \infty$, we conclude that the identities in Theorem 2.1 are satisfied for all test fucntions $\boldsymbol{\eta} \in L_2(0,T;\boldsymbol{V}_{0m_0})$,

$\xi \in L_2(0,T;V_{m_0})$ and $\zeta \in L_2(0,T;V_{m_0})$. Next, passing to the limit $m_0 \to \infty$, we arrive by density arguments at identities (2.14). Clearly, a priori estimates (2.15) are the consequences of the uniform estimates in Lemmas 3.1–3.4 and the weak convergences (5.2). This proves the theorem. □

6. Proof of Theorem 2.2

From Lemmas 3.1–3.4 and 4.1–4.4 it follows that there exists a triple $(\boldsymbol{u}, \chi, \mu)$ with

$$
\begin{aligned}
&\boldsymbol{u} \in L_\infty(0,T;\boldsymbol{H}^2(\Omega)\cap \boldsymbol{H}_0^1(\Omega)), \quad \boldsymbol{u}_t \in L_\infty(0,T;\boldsymbol{H}^1(\Omega)),\\
&\boldsymbol{u}_{tt} \in L_\infty(0,T;\boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_{ttt} \in L_2(0,T;(\boldsymbol{H}_0^1(\Omega))'),\\
&\chi \in C^{1/2}([0,T];H_N^2(\Omega)), \quad \chi_t \in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;H_N^2(\Omega)),\\
&\chi_{tt} \in L_2(0,T;(H_N^2(\Omega))'),\\
&\mu \in L_\infty(0,T;H_N^2(\Omega)), \quad \mu_t \in L_2(\Omega^T),
\end{aligned} \tag{6.1}
$$

and a subsequence of solutions $(\boldsymbol{u}^m, \chi^m, \mu^m)$ to problem (3.6) (which we still denote by the same indices) such that as $m \to \infty$:

$$
\begin{aligned}
&\boldsymbol{u}^m \to \boldsymbol{u} && \text{weakly}-^* \text{ in } L_\infty(0,T;\boldsymbol{H}^2(\Omega)),\\
&\boldsymbol{u}_t^m \to \boldsymbol{u}_t && \text{weakly}-^* \text{ in } L_\infty(0,T;\boldsymbol{H}^1(\Omega)),\\
&\boldsymbol{u}_{tt}^m \to \boldsymbol{u}_{tt} && \text{weakly}-^* \text{ in } L_\infty(0,T;\boldsymbol{L}_2(\Omega)),\\
&\boldsymbol{u}_{ttt}^m \to \boldsymbol{u}_{ttt} && \text{weakly in } L_2(0,T;(\boldsymbol{H}_0^1(\Omega))'),\\
&\chi^m \to \chi && \text{weakly}-^* \text{ in } L_\infty(0,T;H_N^2(\Omega)),\\
&\chi_t^m \to \chi_t && \text{weakly}-^* \text{ in } L_\infty(0,T;L_2(\Omega)) \text{ and}\\
& && \text{weakly in } L_2(0,T;H_N^2(\Omega)),\\
&\chi_{tt}^m \to \chi_{tt} && \text{weakly in } L_2(0,T;(H_N^2(\Omega))'),\\
&\mu^m \to \mu && \text{weakly}-^* \text{ in } L_\infty(0,T;H_N^2(\Omega)),\\
&\mu_t^m \to \mu_t && \text{weakly in } L_2(\Omega).
\end{aligned} \tag{6.2}
$$

Using the compactness results [Sim87] it follows that for a subsequence (still denoted by the same indices)

$$
\begin{aligned}
&\boldsymbol{u}^m \to \boldsymbol{u} && \text{strongly in } L_2(0,T;\boldsymbol{H}_0^1(\Omega))\cap C([0,T[;\boldsymbol{L}_2(\Omega))\\
& && \text{and a.e. in } \Omega^T,\\
&\boldsymbol{u}_t^m \to \boldsymbol{u}_t && \text{strongly in } \boldsymbol{L}_2(\Omega^T)\cap C([0,T];\boldsymbol{L}_2(\Omega))\\
& && \text{and a.e. in } \Omega^T,\\
&\boldsymbol{u}_{tt}^m \to \boldsymbol{u}_{tt} && \text{strongly in } C([0,T];(\boldsymbol{H}_0^1(\Omega))'),\\
&\chi^m \to \chi && \text{strongly in } L_2(0,T;H^1(\Omega))\cap C([0,T];H^1(\Omega))\\
& && \text{and a.e. in } \Omega^T,
\end{aligned} \tag{6.3}
$$

$$\chi_t^m \to \chi_t \quad \text{strongly in } \ L_2(0,T;H^1(\Omega)) \cap C([0,T];(H_N^2(\Omega))') \\ \text{and a.e. in } \ \Omega^T,$$

$$\mu^m \to \mu \quad \text{strongly in } \ L_2(0,T;H^1(\Omega)) \cap C([0,T];H^1(\Omega)) \\ \text{and a.e. in } \ \Omega^T.$$

Hence,

$$\begin{aligned}
\boldsymbol{u}^m(0) = \boldsymbol{u}_0^m &\to \boldsymbol{u}(0) && \text{strongly in } \ \boldsymbol{H}_0^1(\Omega),\\
\boldsymbol{u}_t^m(0) = \boldsymbol{u}_1^m &\to \boldsymbol{u}_t(0) && \text{strongly in } \ \boldsymbol{L}_2(\Omega),\\
\boldsymbol{u}_{tt}^m(0) = \boldsymbol{u}_2^m &\to \boldsymbol{u}_{tt}(0) && \text{strongly in } \ (\boldsymbol{H}_0^1(\Omega))',\\
\chi^m(0) = \chi_0^m &\to \chi(0) && \text{strongly in } \ H^1(\Omega),\\
\chi_t^m(0) = \chi_1^m &\to \chi_t(0) && \text{strongly in } \ (H_N^2(\Omega))',\\
\mu^m(0) = \mu_0^m &\to \mu(0) && \text{strongly in } \ H^1(\Omega),
\end{aligned}$$

what together with convergences (4.3) implies that

$$\begin{aligned}
&\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \boldsymbol{u}_{tt}(0) = \boldsymbol{u}_2,\\
&\chi(0) = \chi_0, \quad \chi_t(0) = \chi_1, \quad \mu(0) = \mu_0.
\end{aligned} \tag{6.4}$$

The relations (6.1) and (6.4) imply assertion (2.19) of the theorem.

We introduce now the following weak formulation of system (4.1):

$$\begin{aligned}
&\int_0^T \langle \boldsymbol{u}_{ttt}^m, \boldsymbol{\eta}\rangle_{(\boldsymbol{H}_0^1(\Omega))', \boldsymbol{H}_0^1(\Omega)} dt + \int_0^T (A\varepsilon(\boldsymbol{u}_t^m), \varepsilon(\boldsymbol{\eta}))dt\\
&= \int_0^T ([z'(\chi^m)B\nabla\chi^m]_{,t} + \boldsymbol{b}_t, \boldsymbol{\eta})dt \quad \forall \boldsymbol{\eta} \in L_2(0,T;\boldsymbol{V}_{0m}),\\
&\int_0^T \langle \chi_{tt}^m, \xi\rangle_{(H_N^2(\Omega))', H_N^2(\Omega)} = \int_0^T (\mu_t, \Delta\xi)dt \quad \forall \xi \in L_2(0,T;V_m),\\
&\int_0^T (\mu_t^m, \zeta)dt = -\int_0^T (\Delta\chi_t^m, \zeta)dt + \int_0^T ([\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]_{,t}, \zeta)dt\\
&\qquad \forall \zeta \in L_2(0,T;V_m).
\end{aligned} \tag{6.5}$$

We pass to the limit $m \to \infty$ in a similar fashion as in the proof of Theorem 2.1. Clearly, due to the weak convergences (6.2), all linear terms in identities (6.5) converge to the corresponding limits. It remains to examine the convergence of the nonlinear terms $[z'(\chi^m)B\nabla\chi^m]_{,t}$ and $[\psi'(\chi^m) + W_{,\chi}(\varepsilon(\boldsymbol{u}^m), \chi^m)]_{,t}$ whose explicit expressions are given in (2.21).

Recalling assumptions on z, ψ and using Lemmas 3.1, 4.1 we obtain the following bounds (these bounds can be also directly concluded from the proofs of Lemmas 4.3, 4.4):

$$
\begin{aligned}
&\|[z'(\chi^m)B\nabla\chi^m]_{,t}\|_{L_2(\Omega^T)} \le c(\|\chi_t^m\nabla\chi^m\|_{L_2(\Omega^T)} + \|\nabla\chi_t^m\|_{L_2(\Omega^T)})\\
&\quad \le c(\|\chi_t^m\|_{L_2(0,T;L_\infty(\Omega))}\|\nabla\chi^m\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\chi_t^m\|_{L_2(\Omega^T)})\\
&\quad \le c(c_0+1)c_4(T),\\
&\|[\psi'(\chi^m)]_{,t}\|_{L_2(\Omega^T)} \le c\|((\chi^m)^2+1)\chi_t^m\|_{L_2(\Omega^T)}\\
&\quad \le c(\|\chi^m\|^2_{L_\infty(0,T;L_4(\Omega))}\|\chi_t^m\|_{L_2(0,T;L_\infty(\Omega)} + \|\chi_t^m\|_{L_2(\Omega^T)})\\
&\quad \le c(c_1^2+1)c_4(T),\\
&\|[W_{,\chi}(\varepsilon(u^m),\chi^m)]_{,t}\|_{L_2(\Omega^T)}\\
&\quad \le c(\|\chi_t^m\varepsilon(u^m)\|_{L_2(\Omega^T)} + \|\chi_t^m\|_{L_2(\Omega^T)} + \|\varepsilon(u_t^m)\|_{L_2(\Omega^T)})\\
&\quad \le c(\|\chi_t^m\|_{L_2(0,T;L_\infty(\Omega))}\|\varepsilon(u^m)\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi_t^m\|_{L_2(\Omega^T)}\\
&\qquad + \|\varepsilon(u_t^m)\|_{L_2(\Omega^T)}) \le c(c_0+1)c_4(T) + c_5(T).
\end{aligned}
\tag{6.6}
$$

Thanks to these uniform in m bounds and the pointwise convergences (6.3) we can apply the nonlinear convergence lemma (see [Lions69], Chap. 1, Lemma 1.3) to conclude that

$$
\begin{aligned}
&[z'(\chi^m)B\nabla\chi^m]_{,t} = z''(\chi^m)\chi_t^m B\nabla\chi^m + z'(\chi^m)B\nabla\chi_t^m\\
&\quad \to z''(\chi)\chi_t B\nabla\chi + z'(\chi)B\nabla\chi_t = [z'(\chi)B\nabla\chi]_{,t} && \text{weakly in } L_2(\Omega^T),\\
&[\psi'(\chi^m)]_{,t} = (3(\chi^m)^2-1)\chi_t^m \to (3\chi^2-1)\chi_t = [\psi'(\chi)]_{,t} && \text{weakly in } L_2(\Omega^T),\\
&[W_{,\chi}(\varepsilon(u^m),\chi^m)]_{,t} = z''(\chi^m)\chi_t^m(B\cdot\varepsilon(u^m) + Dz(\chi^m) + E)\\
&\quad + z'(\chi^m)(B\cdot\varepsilon(u_t^m) + Dz'(\chi^m)\chi_t^m)\\
&\quad \to z''(\chi)\chi_t(B\cdot\varepsilon(u) + Dz(\chi) + E)\\
&\quad + z'(\chi)(B\cdot\varepsilon(u_t) + Dz'(\chi)\chi_t) = [W_{,\chi}(\varepsilon(u),\chi]_{,t} && \text{weakly in } L_2(\Omega^T).
\end{aligned}
\tag{6.7}
$$

In view of (6.7), passing to the limit $m \to \infty$ in identities (6.5) we conclude (2.20). We also note that a priori estimates (2.22) result immediately from the estimates in Lemmas 3.1–4.3, 4.1–4.4 and the weak convergences (6.2). this completes the proof of the theorem. □

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